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Entanglement of Permutation Symmetric State

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We consider an assembly of N indistinguishable two-level atoms, for which a concept of spin-squeezing and entanglement measures can be well defined. Permutation symmetry of such a state allows one to use an entanglement measure called the concurrence as the measures of pairwise and also N-partite entanglement. By using these measures, we show that negative pair-spin correlations of the state are quantum ones. It implies the quantumness of a spin-squeezing, which also possesses pair-spin negative correlations. We also provide the entanglement measures of a macroscopic entangled state and an atomic Dicke state for super-radiant and spontaneous emissions.

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I. INTRODUCTION

It is well known that entanglement and squeezing are valuable resources of quantum correlations – vital implication of quantum mechanics for multiple particles. A superposition principle and Hilbert space structure jointly imply quantum entanglement, which recently becomes a subject of intense study due to its being indispensable ingredient for quantum information (QI) processing [1]. The uncertainty principle originating from non-commutativity, which is so fundamental that it leads to the indeterminacy and probability in quantum mechanics, lays a quantum limitation in an extraction of classic information from a quantum state. Based on this principle, a squeezed state is defined to be a special kind of entangled state for overcoming such a limit to a certain extent. Thus, it is important to study a relation between squeezing and entanglement, and an actual role of squeezing in QI processing.

Entanglement should be indicated by its measures and many kinds of entanglements are defined [2]. One of the bold results obtained so far is the discovery of the concurrence [3] a computable measure of entanglement of formation and its application gives many interesting results [4]. The finding of similar computable measure for multipartite entanglement is of fundamental importance and remains to be solved.

Regarding to the squeezing, one should consider bosonic and spin cases separately due to the basic difference in their commutation relations. Boson squeezing is well studied and have been used for continuous variable QI processing [5]. As far as for spin case, a concept of spin squeezing was established in [6] based on the idea of quantum correlations and demonstrated with the use of collective nonlinear Hamiltonians. The main idea in the spin squeezing is that quantum correlations are needed to overcome a standard quantum limit (SQL) but not classical correlations. Wineland et. al. have applied the spin squeezing concept to atomic spectroscopic precision improvement over SQL [7] and have given another definition to which we refer as a spectroscopic squeezing. Spin squeezed or atomic squeezed states have been experimentally obtained [8-10].

It is very important to explicitly show the quantumness, i.e., posessing quantum correlations, in the definition in order to use it in a correct sense. In Ref. [11] it has been shown that spectroscopic spin squeezing is a sufficient condition for inseparability or entanglement. But, this result does not necessarily imply the quantumness in the original definition of spin squeezing, because it is weaker in mathematical sense than the spectroscopic squeezing. While demonstrating the ability of a spin squeezed state to improve precision measurement over SQL in Ramsey spectroscopy in the presence of decoherence, we have previously pointed out the negative pair-spin correlations and pairwise entanglement in spin squeezed state [12]. In this paper we deal with this issue in detail.

Let us restate the definition of spin squeezing [6] here: a spin is regarded as squeezed only if the minimum variance $\langle \Delta J_\perp^2 \rangle$ of a spin component per-

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pendicular to the direction $\hat{n}$ of the mean spin vector is smaller than the SQL of $J/2$ of the coherent spin state (CSS), i.e., the state is spin-squeezed iff $\langle \Delta J_\perp^2 \rangle / (J/2) < 1$. There are two physical reasons to intuitively explain why this definition should imply quantum correlations: (1) The use of minimum variance is, in fact, equivalent to the use of Roberston uncertainty inequality [[13]], in which classic correlations or covariance are extracted, instead of usual Heisenberg uncertainty relation. (2) It is defined to be relative to the CSS, thus it is intuitively assumed to be valid for the state of a subspace to which the CSS belongs. A state in this subspace turns out to be a permutation symmetric state or a state of indistinguishable two-level atoms.

In this paper, we prove that the intuition was indeed correct by the use of entanglement measure. In fact, we use the concurrence in the study of quantum correlations of indistinguishable particles, because permutation symmetry of the state allows one to apply it as a pairwise and N-partite entanglement measure [14]. As a main result of the paper, we show that the negative correlations in a permutation symmetric state are quantum ones. Quantumness of spin-squeezing in the case of permutation symmetry is an implication of this result. We also make sure the quantumness of atomic Dicke states [15], which emits spontaneous and stimulated emissions, by finding the amount of entanglement in these states.

Now we proceed with the main part of the paper.

II. PAIRWISE AND N-PARTITE ENTANGLEMENT OF INDISTINGUISHABLE ATOMIC AND PERMUTATION SYMMETRIC STATE

Let us assume that we are given an assembly of N indistinguishable two-level atoms or $\frac{1}{2}$-spins. The indistinguishability means that, the time evolution of the assembly is governed by a Hamiltonian which is left unchanged under any permutation of the atomic labels. Therefore, a permutation symmetric eigenstate

$$|\Phi_{\rm sym}(N,k)\rangle = \sum_{\rm perm} \left|1^{\otimes k} 0^{\otimes(N-k)}\right\rangle = \binom{2J}{k}^{1/2} |J, J-k\rangle, \quad (1)$$

in the generic 0 and 1 basis arises in the description of N identical two-level atoms. Here, $\sum_{\rm perm}$ describes a summation over all distinct permutations of $N-k$ zero's and k one's, and $J(J+1)$ ($J = N/2$) and m are eigenvalues of the operators $\vec{J}^2$ and J_z, respectively. Also, the collective angular momentum operator is defined by $J_k = \frac{1}{2}\sum_{i=1}^{N} \sigma_{k,i}$ ($k = x, y, z$), where $\sigma_{k,i}$ is a Pauli operator for ith atom or spin-1/2.

The angular momentum states $|J,m\rangle$ are sometimes called as atomic Dicke states and form an orthonormal basis $\langle J,m'|J,m\rangle = \delta_{mm'}$ in the maximum $2J+1$ multiplicity subspace of constant J. Therefore, any permutation symmetric pure state is given by

$$|\Psi\rangle = \sum_{k=0}^{2J} a_k |J, J-k\rangle = \sum_{k=0}^{2J} c_k |\Phi_{sym}(N,k)\rangle, \quad (2)$$

where $a_k = \binom{2J}{k}^{1/2} c_k$.

It is well known that an atomic Dicke state $|J,m\rangle$ displays strong atom-atom correlations [15]. There are $2J+1$ subspaces labeled by different m ($m = -J, -J+1, \ldots, J$) and invariant under rotation operators, thus each subspace may have unique correlations. We single out CSS's $|J, \pm J\rangle$ due to their obvious separability, thus possessing only classic correlations. Another interesting state is the super-radiance state ($m \approx 0$), which has no dipole momentum responsible for classical radiation so that its spontaneous emission must have quantum origin. Then one may ask, if the other Dicke states are quantum correlated or if the stimulated emissions of Dicke state imply quantum correlations or multipartite entanglement and we answer them using an entanglement measure.

For the purpose of the paper it is enough to use only one specific measure, namely the concurrence. Concurrence is originally defined as $C_{\rm mix} = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ for a mixed state of a pair of spins AB, where λ_1, λ_2, λ_3, and λ_4 are the square roots of the eigenvalues of the product $\rho_{AB}\tilde{\rho}_{AB}$ with $\tilde{\rho}_{AB} = (\sigma_y \otimes \sigma_y)\rho^*_{AB}(\sigma_y \otimes \sigma_y)$ in decreasing order. Here the asterisk denotes complex conjugation. For a pure state, it reduces simply to $C_{\rm pure} = 2\sqrt{\det \rho_A}$, where $\rho_A = Tr_B\rho_{AB}$.

Although there is no widely accepted definition of concurrence for more qubits, the permutation symmetry of the state allows one to apply the above definition in the following way, as done in Ref. [14], to define:

(1) The entanglement of a spin with the other $(N-1)$ spins is defined by

$$C_{\bf whole} = C_{\bf pure} = 2\sqrt{\det \rho_A} \quad (3)$$

with $\rho_A = Tr_{BCD\ldots}\rho$, i.e., by considering all other spins except spin A as a single object. In fact, it turns out to be N-partite entanglement measure for indistinguishable spins.

(2) Pairwise entanglement is defined by

$$C_{\bf pair} = C_{\rm mix} = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \quad (4)$$

with $\rho_{AB} = Tr_{CDE\ldots}\rho$. We refer to Coffman *et. al.* [14] for details and just note that $C_{\bf whole}$ is valid only for a pure state while $C_{\bf pair}$ may also be used for a mixed state.

For calculating these and other entanglement measures the use of the Clebsch-Gordan composition turns out to be much useful tool, since a permutation symmetric state can be considered as spin-J state. By considering spin-J as composed of spin-J_1 and spin-$(J-J_1)$, one may write the Clebsch-Gordan composition in the following simple form:

$$|\Phi_s(N,k)\rangle = \sum_{l=0}^{\min\{2J_1,k\}} |\Phi_s(2J_1,l)\rangle \otimes |\Phi_s(N-2J_1,k-l)\rangle . \quad (5)$$

For instance, with $J_1 = 1$ we can easily write the reduced density matrix $\rho_{AB} = Tr_{CD...}|\Psi\rangle\langle\Psi|$ as:

$$\rho_{AB} = \begin{pmatrix} \langle\varphi_0|\varphi_0\rangle & \langle\varphi_0|\varphi_1\rangle & \langle\varphi_0|\varphi_1\rangle & \langle\varphi_0|\varphi_2\rangle \\ \langle\varphi_1|\varphi_0\rangle & \langle\varphi_1|\varphi_1\rangle & \langle\varphi_1|\varphi_1\rangle & \langle\varphi_1|\varphi_2\rangle \\ \langle\varphi_1|\varphi_0\rangle & \langle\varphi_1|\varphi_1\rangle & \langle\varphi_1|\varphi_1\rangle & \langle\varphi_1|\varphi_2\rangle \\ \langle\varphi_2|\varphi_0\rangle & \langle\varphi_2|\varphi_1\rangle & \langle\varphi_2|\varphi_1\rangle & \langle\varphi_2|\varphi_2\rangle \end{pmatrix}, \quad (6)$$

where $\langle\varphi_j|\varphi_i\rangle = \sum_{k=0}^{N-2} c_{k+i}c^*_{k+j}\binom{N-2}{k}$ $(i,j = 0,1,2)$, which has at most three eigenvalues due to its symmetry property. Using the following correlation sums

$$\sum_{k=0}^{2J} k\binom{2J}{k} c_k c^*_{k-1} = \langle J_+\rangle, \quad (7)$$

$$\sum_{k=0}^{2J} k(k-1)\binom{2J}{k} c_k c^*_{k-2} = \langle J_+^2\rangle, \quad (8)$$

$$\sum_{k=0}^{2J} k\binom{2J}{k} |c_k|^2 = \langle J - J_z\rangle, \quad (9)$$

$$\sum_{k=0}^{2J} k^2\binom{2J}{k} |c_k|^2 = \langle (J - J_z)^2\rangle, \quad (10)$$

$$\sum_{k=0}^{2J} k^2\binom{2J}{k} c_k c^*_{k-1} = \langle J_+(J-J_z)\rangle = \frac{1}{2}\{(N+1)\langle J_+\rangle - \langle J_+J_z + J_zJ_+\rangle\}, \quad (11)$$

one can express the matrix elements via mean values of the first and second order momenta as:

$$\langle\varphi_0|\varphi_0\rangle = \frac{J(J-1) + \langle J_z^2\rangle + (2J-1)\langle J_z\rangle}{N(N-1)}, \quad (12)$$

$$\langle\varphi_1|\varphi_1\rangle = \frac{J^2 - \langle J_z^2\rangle}{N(N-1)}, \quad (13)$$

$$\langle\varphi_2|\varphi_2\rangle = \frac{J(J-1) + \langle J_z^2\rangle - (2J-1)\langle J_z\rangle}{N(N-1)}, \quad (14)$$

$$\langle\varphi_0|\varphi_1\rangle = \frac{(J-\frac{1}{2})\langle J_+\rangle + \frac{1}{2}\langle J_zJ_+ + J_+J_z\rangle}{N(N-1)}, \quad (15)$$

$$\langle\varphi_0|\varphi_2\rangle = \frac{\langle J_+^2\rangle}{N(N-1)}, \quad (16)$$

$$\langle\varphi_1|\varphi_2\rangle = \frac{(J-\frac{1}{2})\langle J_+\rangle - \frac{1}{2}\langle J_zJ_+ + J_+J_z\rangle}{N(N-1)}. \quad (17)$$

It is obvious that extremums of the momentum mean values are the rotational invariants. Some invariants constructed in order to give zero values for separable states may serve as an entanglement indication or even measure. In fact, as we see later, the definition of concurrence is a procedure to find such an extreme quantity defined by mean values of the first and second order quantum momenta. In a similar way, k-partite correlations sums or k order extreme momenta are useful to define k-partite entanglement measure and this issue will be discussed somewhere else.

Now with Eq. (6) it is easy to see that N-partite entanglement or concurrence C_{whole} of permutation symmetric pure state is defined only by the mean spin value as

$$C_{\text{whole}} = \sqrt{1 - (\langle J_{\hat{n}}\rangle/J)^2}, \quad (18)$$

by using

$$\rho_A = Tr_B\rho_{AB} = \frac{1}{J}\begin{pmatrix} J + \langle J_z\rangle & \langle J_-\rangle \\ \langle J_+\rangle & J - \langle J_z\rangle \end{pmatrix} \quad (19)$$

and $\langle J_+\rangle\langle J_-\rangle = \langle J_x\rangle^2 + \langle J_y\rangle^2$. In other words, the extreme mean of first order momentum $J_{\hat{n}}$ serves as an entanglement measure of a spin with the others or N-partite entanglement in a permutation symmetric pure state. As we noted before, it is valid only for a pure state. This can also be easily shown using the composition Eq. (5) with $J_1 = 1/2$.

As for pairwise entanglement we need to use Eq. (6) in some appropriate frame of reference to simplify the problem. Under the rotation of the frame of reference both entanglement and squeezing should not change. Therefore, by applying appropriate rotations one may choose as: $\langle J_x\rangle = \langle J_y\rangle = 0$, $\langle J_z\rangle = \langle J_{\hat{n}}\rangle$. Then by means of a virtual rotation around $\hat{n}$, the extrema of perpendicular variances are found in the form $\langle J^2_{\substack{\top\\ \perp}}\rangle = \frac{1}{2}\{\langle J_x^2 + J_y^2\rangle \pm \sqrt{[\langle J_x^2 - J_y^2\rangle]^2 + \langle J_xJ_y + J_yJ_x\rangle^2}\}$, or we have $\langle J_x^2 + J_y^2\rangle = \langle J_\top^2 + J_\perp^2\rangle$ and $|\langle J_+^2\rangle| = \langle J_\top^2 - J_\perp^2\rangle$, where $\perp$ states for minimum variance perpendicular to the mean spin vector $\hat{n}$ and $\top$ for maximum one. And finally one may apply the Eq. (6).

In the first, let us consider the case of $\langle J_kJ_l + J_lJ_k\rangle = 0$ in detail, which includes also a state symmetric under the exchange of 0 and 1. If we denote $A = (J - 2\langle J_\perp^2\rangle)/[J(2J-1)]$ and $B = \{J^2 - \langle J_{\hat{n}}^2\rangle - [\langle J_\top^2 - J_\perp^2\rangle^2 + 4(J^2 - \langle J_\top^2\rangle)(J^2 - \langle J_\perp^2\rangle)]^{1/2}\}/[J(2J-1)]$, then the concurrence simply reads as:

$$C_{\text{pair}} = \max\{0, |A-B| - (A+B)\} = \begin{cases} \max\{-2A, 0\} & \text{when } A < B \\ \max\{-2B, 0\} & \text{when } B < A \\ |A-B| & \text{when } A+B = 0. \end{cases} \quad (20)$$

On the other side, one may readily check that $A < 0$ if and only if $K^{i,j}_{\perp,\perp} < 0$, and $B < 0$ if and only if $K^{i,j}_{\hat{n},\hat{n}} < 0$ by using pair-spin correlation coefficient $K^{i,j}_{k,l} = \frac{1}{2}[\langle\sigma_{\mu,i}\sigma_{\nu,j} + \sigma_{\nu,i}\sigma_{\mu,j}\rangle -$

$(\langle\sigma_{\mu,i}\rangle\langle\sigma_{\nu,j}\rangle + \langle\sigma_{\nu,i}\rangle\langle\sigma_{\mu,j}\rangle)]$ as follows. In the case of indistinguishable spins, one easily finds negativity conditions as $K^{ij}_{\mu\mu} = \frac{1}{N-1}[2\langle\Delta J_\mu^2\rangle/J - (1 - \langle J_\mu\rangle^2/J^2] < 0$ and $K^{ij}_{\mu\nu} = \frac{2}{N-1}[\langle J_\mu J_\nu + J_\nu J_\mu\rangle/J - (N-1)\langle J_\mu\rangle\langle J_\nu\rangle/J^2] < 0$. In the principal coordinate system these conditions become of the form

$$\langle\Delta J_\perp^2\rangle/(J/2) < 1 \quad \text{or} \quad A < 0, \quad (21)$$

$$2J\langle\Delta J_{\vec{n}}^2\rangle/(J^2 - \langle J_{\vec{n}}\rangle^2) < 1, \quad \text{or} \quad B < 0, \quad (22)$$

$$\langle J_{\vec{n}}J_k + J_k J_{\vec{n}}\rangle < 0 \quad (k = \perp, \top). \quad (23)$$

Thus, we see the equivalence between the negative pair-spin correlation ($K^{i,j}_{\mu,\mu} < 0$) and quantum pairwise entanglement ($C_{\text{pair}} > 0$) in the case of $\langle J_k J_l + J_k J_k\rangle = 0$ symmetry. For instance, the following simple relation holds $C_{\text{pair}} = -K^{i,j}_{\perp,\perp} = (1-\xi_\perp)/(N-1)$ for spin-squeezed state $\xi_\perp < 1$. The state with $A + B = 0$ is outlined in Eq. (20) because in this case an equality holds for the following nontrivial inequality

$$(J^2 - \langle J_\perp^2\rangle)(J^2 - \langle J_\top^2\rangle) \geq \frac{1}{4}(N-1)^2\langle J_{\vec{n}}\rangle^2 \quad (24)$$

and its concurrence becomes

$$C_{\text{pair}} = (|1-\xi_\perp|)/(N-1).$$

For a general case, an analysis using solution for C_{pair} may be complicated, thus we proceed further using the following simple observation; It may be easily shown that the criteria $C_{\text{pair}} > 0$ or a state being pairwise entangled is equivalent to the condition $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) > 0$ in a symmetric form, when $\rho_{AB}\tilde{\rho}_{AB}$ has at most three eigenvalues as in our permutation symmetric case. The secular equation of Eq. (6) becomes cubic $\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$ due to its symmetry. Using Newton's identities $\lambda_1 + \lambda_2 + \lambda_3 = -a_2$ and $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = a_2$ one gets a simple criteria for pairwise entanglement as $a_2^2 - 4a_1 > 0$ without the need of solving the equation to find λ_i. This criteria may be written in the final form as:

$$\begin{aligned}&(2\langle J_\perp^2\rangle/J - 1)(\langle J_z J_\top + J_\top J_z\rangle/J)^2\\ &+(2\langle J_\top^2\rangle/J - 1)(\langle J_z J_\perp + J_\perp J_z\rangle/J)^2\\ &+(2\langle J_\perp^2\rangle/J - 1)(2\langle J_\top^2\rangle/J - 1)\\ &\times[1 - (\langle J_{\vec{n}}\rangle/J)^2 - 2\langle\Delta J_{\vec{n}}^2\rangle/J] > 0. \quad (25)\end{aligned}$$

Using the following nontrivial uncertainty inequality

$$\langle J_\perp^2\rangle + \frac{\langle\Delta J_{\vec{n}}^2\rangle}{1 - (\langle J_{\vec{n}}\rangle/J)^2} \geq J \quad (26)$$

and other trivial inequalities [16] including Eq. (24) one may show that the following condition is sufficient one for $C_{\text{pair}} > 0$, i.e., the state to be entangled:

$$\begin{aligned}&(2\langle J_\perp^2\rangle/J - 1)(2\langle J_\top^2\rangle/J - 1)\times\\ &[1 - (\langle J_{\vec{n}}\rangle/J)^2 - 2\langle\Delta J_{\vec{n}}^2\rangle/J] > 0. \quad (27)\end{aligned}$$

Finally, inequality $\langle J_\top^2\rangle + \langle J_\perp^2\rangle \geq J$ with Eqs. (21, 22,27) says the main result of the paper: negativity of pair-spin correlations are sufficient condition for pairwise entanglement in an assembly of indistinguishable spins or atoms.

One implication of this result is that any spin-squeezed state $\xi_\perp = \langle\Delta J_\perp^2\rangle/(J/2) < 1$ is quantum entangled and possesses negative pair-spin correlations. Such a state is able to overcome SQL even in the presence of decoherence as shown in Ref. [12]. As for spectroscopic spin squeezing defined as $\xi_R = \xi_\perp J^2/\langle J_{\vec{n}}^2\rangle$ we see that more pairwise entanglement and less N-partite entanglement are needed to have such a property for a pure state case. It is interesting if truly N-partite entanglement exists in the case of a mixed state. Finally let us apply the results to some simple but interesting cases.

III. ENTANGLEMENT OF SOME INTERESTING STATES

Example 1. *Dicke state* $|J,m\rangle$. It is well known that the so called Dicke states $|J,m\rangle$ displays strong atom-atom correlations. Experimental fact for quantum correlations is that, Dicke states, having no quadruple momentum responsible for classical radiation, exhibit the super-radiance in spontaneous emission with quantum origin. Exact quantities of entanglements for a Dicke state $|J,m\rangle$ are $C^2_{\text{whole}} = (J+m)(J-m)/J^2$ and

$$C_{\text{pair}} = \frac{\sqrt{(J+m)(J-m)}}{J(2J-1)}\Big[\sqrt{(J+m)(J-m)} - \sqrt{(J+m-1)(J-m-1)}\Big],$$

and we see that the states with $m \approx 0$ for the superradiance is indeed maximally entangled (See Fig. 1). ($C_A = 1$ for the state $|J,0\rangle$ and $C_A = (N^2-1)/N^2$ for the state $|J,1/2\rangle$ depending even and odd number of particles.) Atomic coherent state $|J,\pm J\rangle$ has no quantum correlation $C_{\text{whole}} = C_{\text{pair}} = 0$ although it has maximum classical correlation. All states except coherent spin state ($m = \pm J$) always have pairwise entanglement greater than $\frac{1}{N}$ — the half of possible maximum entanglement $\frac{2}{N}$. Dicke states do not possess negative correlations and spin-squeezing so that $\langle J^2_{max}\rangle = \langle J^2_{min}\rangle \geq J/2$.

Example 2. *Generalized Schrödinger cat-like state.* It is interesting to consider a symmetric superposition of Dicke states $|\varphi\rangle = \frac{1}{\sqrt{2}}(|J,m\rangle + |J,-m\rangle)$. These states are maximally entangled in the sense that $C_{\text{whole}} = 1$. Pair-wise concurrence is $C_{\text{pair}} = \max\left\{0, \frac{(J-2m^2)}{j(2j-1)}\right\}$ and only the states with $m \in [-\sqrt{N}/2, \sqrt{N}/2]$ have pair-wise entanglement, while the others close to the n-Cat state $\frac{1}{\sqrt{2}}(|J,J\rangle + |J,-J\rangle)$ do not (See Fig. 2).

Thus, a percent of the states having pair-wise entanglement among the states with different m goes to $1/\sqrt{N}$ for large N. It coincides with the best possible spectroscopic and interferometric improvement over the standard quantum limit. Maximum pair-wise entanglement is never exceeds $1/(N-1)$. It is also interesting that these states are negatively correlated having $\langle \Delta J_{\vec{n}}^2 \rangle = 0$, but not spin-squeezed: $\langle J_\perp^2 \rangle \geq J/2$.

Example 3. *Equal probability superposition of Dicke states.* For equal probability superposition of Dicke states with $a_k = \frac{1}{\sqrt{N+1}}$ $(k = 0, \ldots, N)$, we have $1/3 \geq C_{\mathrm{whole}} < \sqrt{1-\pi^2/16}$ (See Fig. 3). Multipartite entanglement reaches to $C_{\mathrm{whole}} = \sqrt{1-\pi^2/16} = 0.61899$ in the limit $N \to \infty$.

Example 4. *Spin-squeezing models.* To illustrate the spin-squeezing concept in Ref [6], the following two nonlinear Hamiltonians are considered: One-axis twisting $H_{\mathrm{oa}} = \hbar\chi J_z^2$ and two-axis twisting $H_{\mathrm{ta}} = \hbar\chi(J_+^2 - J_-^2)$ with initial CSS. Here $\nu = \chi t$ is a squeezing evolution parameter and t is an evolution time. These nonlinear Hamiltonians themselves are deeply involved in Bose-Einstein condensation (BEC) [11]. For both models, we have found one-to-one correspondence $\xi_\perp = 1 - (N-1)C_{\mathrm{pair}}$, except the case of $N = 3$ for two-axis twisting for which $C_{\mathrm{pair}} = \frac{|1-\xi_\perp|}{N-1}$. Entanglement evolutions in these models are given in Fig. 4-6.

IV. CONCLUSION

In this paper, we have considered entanglement of indistinguishable atoms by introducing new measures of N-partite and pairwise entanglement, and given their explicit expressions via experimentally measurable values of collective operators. We have touched some physical motivation for concurrence [3, 14] and its possible extention for multipartite case. We have proved that any spin squeezing [6] $\langle \Delta J_\perp^2 \rangle/(J/2) < 1$ implies quantum, and also, negative pairwise correlations, and shown that mean spin value may serve as an N-partite entanglement measure for a pure state. Entanglement of some interesting states including atomic Dicke states and extended cat states are studied.

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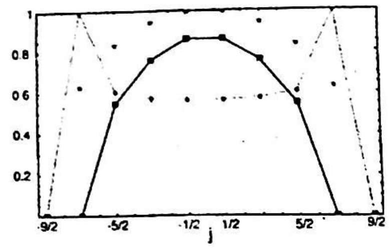

FIG. 1: Entanglement measures for a Dicke state $|J, m\rangle$. All states except CSS are pair-wise and also multi-partite entangled. C_{whole} and τ are increased and C_{pair} is decreased, but always exceeding $1/N$ for small m. The increasing of entanglement is in good agreement with quantum super-radiance of Dicke state with $m \approx 0$

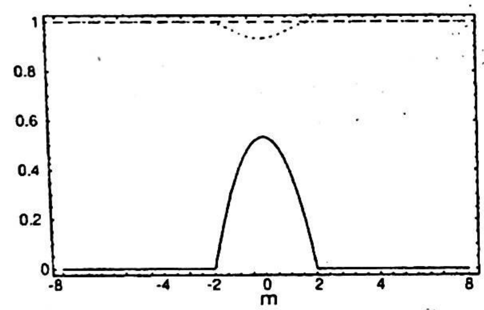

FIG. 2: Pair-wise concurrence for a symmetric superposition state $|J, m\rangle + |J, -m\rangle$. All these states have $C_{\mathrm{whole}} = 1$, thus are maximally entangled. It is interesting that the states in the interval $m \in [-\sqrt{N}/2, \sqrt{N}/2]$ have pair-wise entanglement and the other states neighbouring N-Cat state $|J, J\rangle + |J, -J\rangle$ have non pair-wise entanglement.

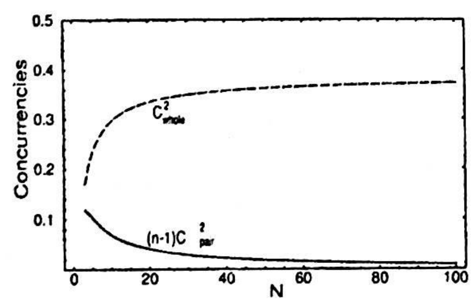

FIG. 3: Equal probability superposition of Dicke states $|\Psi\rangle = \frac{1}{\sqrt{N+1}} \sum_{m=-J}^{J} |J, m\rangle$. The limit of entanglement C_{whole} is $\sqrt{1 - \pi^2/16} = 0.392699$.

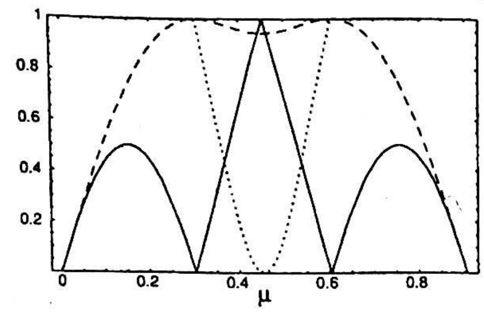

FIG. 4: Entanglements of two-axis twisting vs. squeezing parameter Interesting case of non-symmetric under the exchange ground and exited states. Concurrence is $C_{\mathrm{pair}} = \frac{|J-2\langle J^2_{min}\rangle|}{n(n-1)}$ although $\langle J^2_{max}\rangle \neq \langle J^2_{min}\rangle$. It has also negative correlations, even the state is not spin-squeezed $\langle J^2_{min}\rangle \geq J/2$. For this regime another entanglement measure "negativity" is not coincides with concurrence.

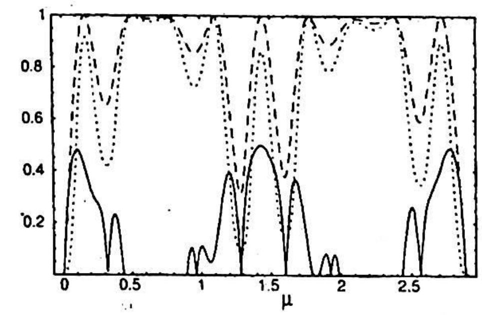

FIG. 5: The "negativity" measure of entanglement coincides with concurrence. As one can see in this complicated evolution one can see all the features discussed in the other figures. For instance there is a point where both entanglement measures are maximal.

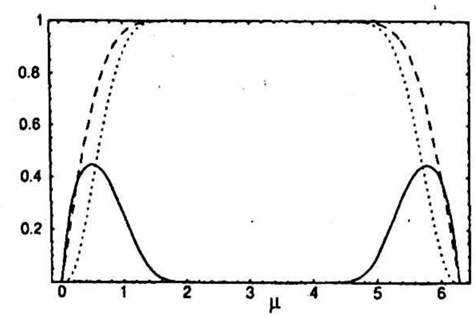

FIG. 6: Entanglements of one-axis twisting vs. squeezing parameter For small squeezing there appears pair-wise entanglement, although the state remains most of the time with maximally entangled states having no pair-wise entanglement when increasing N.