New Methods to Taking Fractional Derivatives and Its Applications

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Abstract

Recently, interest in fractional calculus has been encouraged by its applications in the different fields of science. The applications of fractional calculus include stochastic processes, anomalous diffusion and rheology. In this paper we present new formulas for taking fractional derivatives and some considerations of its applications for the harmonic oscillator.

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I. INTRODUCTION

In the 17th century S.I.Newton and G.W.Leibnitz were developing integral and differential calculations which are Mathematical forms that define motions and patterns of nature. In the 1695 the derivative of order $\alpha = \frac{1}{2}$ was described by Leibnitz in his letter to L'hospital [6, 7, 8, 9]. Recently fractional derivatives have played an important role in mathematical methods and their physical and chemical applications [4, 5, 6]. Various type of fractional derivatives were studied: Riemann-Liouville, Caputo, Hadamard, Erdélyi-Kober, Grünwald-Letnikov, Marchand and Riesz are just a few to name [6, 10, 11, 12, 13, 14]. The most usual formula for taking fractional derivatives is the Riemann-Liouville formula

$$\left(\frac{d}{dx}\right)^{\nu} F(x) = \frac{1}{\Gamma(-\nu)} \int\limits_0^\infty dt F(t)(x-t)^{-1-\nu}.$$

Here

$$Re\nu < 0.$$

It is possible to find a solution of fractional derivatives, but there are some problems. Because Gamma function $\Gamma(x)$ takes infinity value at the point $(0, -1, -2, ...)$. So we study fractional derivatives by means of infinite integer-order differentials which allow us to derive some universal formulas for taking fractional derivatives for wide classes of functions.

We used two general formulas to derive new universal formulas.
The first general formula [1]

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} F(x) = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right)} \int\limits_0^\infty dt \cdot e^{-t^{\nu}\frac{d}{dx}} \cdot F(x). \quad (1)$$

The second general formula [1]

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \cdot F(x) = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right)\sin\frac{\pi}{2\nu}} \int\limits_0^\infty dt \times$$

$$\sin\left(t^{\nu}\frac{d}{dx}\right) F(x). \quad (2)$$

II. USEFUL FORMULAS

The following formulas were used:

$$\exp\left[-t^{\nu}\frac{d}{dx}\right] = 1 - t^{\nu}\frac{d}{dx} + \frac{1}{2!}t^{2\nu}\frac{d^2}{dx^2} - \dots, \quad (3)$$

$$\sin\left(t\frac{d}{dx}\right) = t\frac{d}{dx} - \frac{1}{3!}t^3\frac{d^3}{dx^3} + \frac{1}{5!}t^5\frac{d^5}{dx^5}\dots, \quad (4)$$

$$\cos\left(t\frac{d}{dx}\right) = 1 - \frac{1}{2!}t^2\frac{d^2}{dx^2} + \frac{1}{4!}t^4\frac{d^4}{dx^4}\dots, \quad (5)$$

the following formulas (6, 7) adopted from [2]

$$\int\limits_0^\infty cost^{\nu}dt = \frac{\Gamma\left(\frac{1}{\nu}\right)}{\nu} \cdot \cos\left(\frac{\pi}{2\nu}\right), \quad (6)$$

$$\int\limits_0^\infty \sin t^{\nu}dt = \frac{\Gamma\left(\frac{1}{\nu}\right)}{\nu} \cdot \sin\left(\frac{\pi}{2\nu}\right), \quad (7)$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}. \quad (8)$$

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Formula (8) was transformed to the complex plane ξ in the integral form [2] as shown:

$$e^{-x} = \frac{1}{2i} \int\limits_{-\beta+i\infty}^{-\beta-i\infty} d\xi \frac{x^{\xi}}{\sin(\pi\xi)\cdot\Gamma(1+\xi)} \quad (9)$$

here $-1 < \beta < 0$.

III. DERIVATION OF THE NEW FORMULAS

A. $F(x) = \sin x$

Let $F(x) = \sin x$, using the first general formula (1) given as

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \sin x = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right)} \int\limits_{0}^{\infty} dt \cdot e^{-t^{\nu}\frac{d}{dx}} \cdot \sin x. \quad (10)$$

By using above formula (3) and simple calculations given as

$$e^{-t^{\nu}\frac{d}{dx}} = 1 - t^{\nu}\frac{d}{dx} + t^{2\nu}\frac{d^2}{dx^2\cdot 2!} - t^{3\nu}\frac{d^3}{dx^3\cdot 3!} + t^{4\nu}\frac{d^4}{dx^4\cdot 4!} - \cdots .$$

Substituting above decomposition in equation (10) is the following form

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \sin x = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right)} \int\limits_{0}^{\infty} dt \times \sin x \left(1 - t^{\nu}\frac{d}{dx} + t^{2\nu}\frac{d^2}{dx^2\cdot 2!} - t^{3\nu}\frac{d^3}{dx^3\cdot 3!} + t^{4\nu}\frac{d^4}{dx^4\cdot 4!} - t^{5\nu}\frac{d^5}{dx^5\cdot 5!} + \cdots\right). \quad (11)$$

Let's calculate the derivatives of $\sin x$ function as shown as

$$\sin' x = \cos x \quad \sin'' x = -\sin x \quad \sin''' x = -\cos x$$
$$\sin'''' x = \sin x \quad \sin''''' x = \cos x \quad \cdots .$$

Now substituting above derivatives of $\sin x$ function in formula (4) given as

$$\sin x - t^{\nu}\cos x - t^{2\nu}\sin x\cdot\frac{1}{2!} + t^{3\nu}\cos x\cdot\frac{1}{3!} + t^{4\nu}\sin x\cdot\frac{1}{4!} - t^{5\nu}\cos x\cdot\frac{1}{5!} + \cdots .$$

Let's do some reaggregation for $\sin x, \cos x$ given as the following form

$$\sin x\left(1 - t^{2\nu}\cdot\frac{1}{2!} + t^{4\nu}\cdot\frac{1}{4!} - \cdots\right) - \cos x\left(t^{\nu} - t^{3\nu}\cdot\frac{1}{3!} + t^{5\nu}\cdot\frac{1}{5!} - \cdots\right).$$

Here

$$\left(1 - t^{2\nu}\cdot\frac{1}{2!} + t^{4\nu}\cdot\frac{1}{4!} - \cdots\right) = \cos t^{\nu}$$

and

$$\left(t^{\nu} - t^{3\nu}\cdot\frac{1}{3!} + t^{5\nu}\cdot\frac{1}{5!} - \cdots\right) = \sin t^{\nu}$$

after some transformations and calculations, equation (11) as shown below

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \sin x = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right)} \left[\sin x \int\limits_{0}^{\infty} \cos t^{\nu} dt - \cos x \int\limits_{0}^{\infty} \sin t^{\nu} dt\right]. \quad (12)$$

By using above formula (6) and (7) simplifies to

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \sin x = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right)} \left[\sin x\cdot\frac{\Gamma\left(\frac{1}{\nu}\right)}{\nu} \times \cos\left(\frac{\pi}{2\nu}\right) - \cos x\cdot\frac{\Gamma\left(\frac{1}{\nu}\right)}{\nu}\cdot\sin\left(\frac{\pi}{2\nu}\right)\right] \quad (13)$$

and let's eliminate operations given as the following form

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \sin x = \sin x\cdot\cos\frac{\pi}{2\nu} - \cos x\cdot\sin\frac{\pi}{2\nu}. \quad (14)$$

From the equation (10) we obtain a universal formula as show in (15)

$$\boxed{\left(\frac{d}{dx}\right)^{\rho} \sin x = \sin\left(x + \frac{\pi}{2}\rho\right)} \quad (15)$$

here ρ -is arbitrary order.

B. $F(x) = \cos x$

Note that $F(x) = \cos x$, then using the first general formula (1) and to calculate same as (A) shown as

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \cos x = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right)} \int_0^\infty dt \cdot e^{-t^\nu \frac{d}{dx}} \cdot \cos x$$

$$= \cos\left(x - \frac{\pi}{2\nu}\right). \quad (16)$$

From equation (16) we obtain a universal formula as show in (17)

$$\boxed{\left(\frac{d}{dx}\right)^{\rho} \cos x = \cos\left(x + \frac{\pi}{2}\rho\right)} \quad (17)$$

here ρ -is arbitrary order.

C. Examples

1. $\nu = 1$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

2. $\nu = -1$

$$\left(\frac{d}{dx}\right)^{-1} \sin x = -\cos x = \int \sin(x) dx$$

$$\left(\frac{d}{dx}\right)^{-1} \cos x = \sin x = \int \cos(x) dx$$

3. $\nu = \frac{1}{2}$

$$\left(\frac{d}{dx}\right)^{\frac{1}{2}} \sin x = \frac{\sqrt{2}}{2}(\sin x + \cos x)$$

$$\left(\frac{d}{dx}\right)^{\frac{1}{2}} \cos x = \frac{\sqrt{2}}{2}(\cos x - \sin x)$$

4. $\nu = -\frac{1}{2}$

$$\left(\frac{d}{dx}\right)^{-\frac{1}{2}} \sin x = \frac{\sqrt{2}}{2}(\sin x - \cos x)$$

$$\left(\frac{d}{dx}\right)^{-\frac{1}{2}} \cos x = \frac{\sqrt{2}}{2}(\cos x + \sin x)$$

5. $\nu = \frac{3}{2}$

$$\left(\frac{d}{dx}\right)^{\frac{3}{2}} \sin x = \frac{\sqrt{2}}{2}(\cos x - \sin x)$$

$$\left(\frac{d}{dx}\right)^{\frac{3}{2}} \cos x = \frac{\sqrt{2}}{2}(-\cos x - \sin x)$$

6. $\nu = -\frac{3}{2}$

$$\left(\frac{d}{dx}\right)^{-\frac{3}{2}} \sin x = -\frac{\sqrt{2}}{2}(\sin x + \cos x)$$

$$\left(\frac{d}{dx}\right)^{-\frac{3}{2}} \cos x = \frac{\sqrt{2}}{2}(\sin x - \cos x)$$

7. $\nu = \frac{1}{4}$

$$\left(\frac{d}{dx}\right)^{\frac{1}{4}} \sin x = \sin x \cdot \cos\frac{\pi}{8} + \cos x \cdot \sin\frac{\pi}{8}$$

$$\left(\frac{d}{dx}\right)^{\frac{1}{4}} \cos x = \cos x \cdot \cos\frac{\pi}{8} - \sin x \cdot \sin\frac{\pi}{8}$$

8. $\nu = -\frac{1}{4}$

$$\left(\frac{d}{dx}\right)^{-\frac{1}{4}} \sin x = \sin x \cdot \cos\frac{\pi}{8} - \cos x \cdot \sin\frac{\pi}{8}$$

$$\left(\frac{d}{dx}\right)^{-\frac{1}{4}} \cos x = \cos x \cdot \cos\frac{\pi}{8} + \sin x \cdot \sin\frac{\pi}{8}$$

D. Properties of the Fractional Derivatives

$$\frac{d}{dx}\left(\frac{d}{dx}\right)^{-1} \equiv \left(\frac{d}{dx}\right)^{-1}\left[\frac{d}{dx}\right]$$

$$\frac{d}{dx} \cdot \left(\frac{d}{dx}\right)^{-\frac{1}{2}} \equiv \left(\frac{d}{dx}\right)^{\frac{1}{2}}$$

$$\left(\frac{d}{dx}\right)^{\frac{1}{4}} \cdot \left(\frac{d}{dx}\right)^{\frac{1}{4}} \cdot \sin x \equiv \left(\frac{d}{dx}\right)^{\frac{1}{2}} \sin x$$

$$\left(\frac{d}{dx}\right)^{\frac{1}{4}} \cdot \left(\frac{d}{dx}\right)^{\frac{1}{4}} \cdot \cos x \equiv \left(\frac{d}{dx}\right)^{\frac{1}{2}} \cos x$$

$$\left(\frac{d}{dx}\right)^{-\frac{1}{4}} \cdot \left(\frac{d}{dx}\right)^{-\frac{1}{4}} \cdot \sin x \equiv \left(\frac{d}{dx}\right)^{-\frac{1}{2}} \sin x$$

$$\left(\frac{d}{dx}\right)^{-\frac{1}{4}} \cdot \left(\frac{d}{dx}\right)^{-\frac{1}{4}} \cdot \cos x \equiv \left(\frac{d}{dx}\right)^{-\frac{1}{2}} \cos x$$

E. $F(x) = \sin ax$

Note that $F(x) = \sin ax$, then using the first general formula (1) and to calculate same as (A) shown as

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \sin ax = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right)} \int_0^\infty dt \cdot e^{-t^\nu \frac{d}{dx}} \cdot \sin ax. \quad (18)$$

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \sin ax = a^{-\frac{1}{\nu}} \sin\left(ax - \frac{\pi}{2\nu}\right).$$

From equation (18) we obtain a universal formula as show in (19)

$$\boxed{\left(\frac{d}{dx}\right)^{\rho}\sin ax = a^{\rho}\sin\left(ax+\frac{\pi}{2}\rho\right)} \tag{19}$$

here ρ -is arbitrary order.

F. $F(x) = \cos ax$

Note that $F(x) = \cos ax$, then using the first general formula (1) and to calculate same as (A) shown as

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}}\cos ax = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right)}\int_0^{\infty} dt\cdot e^{-t^{\nu}\frac{d}{dx}}\cdot\cos ax. \tag{20}$$

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}}\cos ax = a^{-\frac{1}{\nu}}\cos\left(ax-\frac{\pi}{2\nu}\right).$$

From equation (20) we obtain a universal formula as show in (21)

$$\boxed{\left(\frac{d}{dx}\right)^{\rho}\sin ax = a^{\rho}\cos\left(ax+\frac{\pi}{2}\rho\right)} \tag{21}$$

here ρ -is arbitrary order.

G. $F(x) = e^{ax}$

Note that $F(x) = e^{ax}$, then using the second general formula (2) and to calculate same as (A) shown as

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}}\cdot e^{ax} = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right)\sin\frac{\pi}{2\nu}}\int_0^{\infty} dt\sin\left(t^{\nu}\frac{d}{dx}\right)e^{ax}. \tag{22}$$

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}}\cdot e^{ax} = a^{-\frac{1}{\nu}}e^{ax}.$$

From equation (22) we obtain a universal formula as show in (23)

$$\boxed{\left(\frac{d}{dx}\right)^{\rho}e^{ax} = a^{\rho}e^{ax}} \tag{23}$$

here ρ -is arbitrary order.

H. $F(x) = e^{-ax}$

Note that $F(x) = e^{-ax}$, then using the second general formula (2) and to calculate same as (A) shown as

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}}\cdot e^{-ax} = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right)\sin\frac{\pi}{2\nu}}\int_0^{\infty} dt\sin\left(t^{\nu}\frac{d}{dx}\right)e^{-ax}. \tag{24}$$

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}}\cdot e^{-ax} = -a^{-\frac{1}{\nu}}e^{-ax}.$$

From equation (24) we obtain a universal formula as show in (25)

$$\boxed{\left(\frac{d}{dx}\right)^{\rho}e^{-ax} = -a^{\rho}e^{-ax}} \tag{25}$$

here ρ -is arbitrary order.

I. $F(x) = a^x$

Note that $F(x) = a^x$, then using the second general formula (2) and to calculate same as (A) shown as

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}}\cdot a^{x} = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right)\sin\frac{\pi}{2\nu}}\int_0^{\infty} dt\cdot\sin\left(t^{\nu}\frac{d}{dx}\right)a^{x}. \tag{26}$$

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}}\cdot a^{x} = (\ln a)^{-\frac{1}{\nu}}a^{x}, \quad a>1.$$

From equation (26) we obtain a universal formula as show in (27)

$$\boxed{\left(\frac{d}{dx}\right)^{\rho}a^{x} = (\ln a)^{\rho}a^{x}, \qquad a>1} \tag{27}$$

here ρ -is arbitrary order.

J. $F(x) = \mathrm{sh}\, ax$

Note that $F(x) = \mathrm{sh}\, ax$, then using the second general formula (2) and to calculate same as (A) shown as

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}}\cdot \mathrm{sh}\, ax = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right)\sin\frac{\pi}{2\nu}}\int_0^{\infty} dt\times$$

$$\sin\left(t^{\nu}\frac{d}{dx}\right)\mathrm{sh}\, ax = a^{-\frac{1}{\nu}}\,\mathrm{ch}\, ax. \tag{28}$$

From equation (28) we obtain a universal formula as show in (29)

$$\boxed{\left(\frac{d}{dx}\right)^{\rho}\mathrm{sh}\, ax = a^{\rho}\,\mathrm{ch}\, ax} \tag{29}$$

here ρ -is arbitrary order.

K. $F(x) = \mathrm{ch}\, ax$

Note that $F(x) = \mathrm{ch}\, ax$, then using the second general formula (2) and to calculate same as (A) shown as

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \cdot \mathrm{ch}\, ax = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right) \sin\frac{\pi}{2\nu}} \int_0^\infty dt \times$$

$$\sin\left(t^\nu \frac{d}{dx}\right) \mathrm{ch}\, ax = a^{-\frac{1}{\nu}}\, \mathrm{sh}\, ax. \qquad (30)$$

From equation (30) we obtain a universal formula as show in (31)

$$\boxed{\left(\frac{d}{dx}\right)^\rho \mathrm{ch}\, ax = a^\rho\, \mathrm{sh}\, ax} \qquad (31)$$

here ρ -is arbitrary order.

L. $F(x) = \frac{1}{x}$

Note that $F(x) = \frac{1}{x}$, substituting in formula (2) is the following form

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \cdot \frac{1}{x} = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right) \sin\frac{\pi}{2\nu}} \int_0^\infty dt \sin\left(t^\nu \frac{d}{dx}\right) \frac{1}{x}. \qquad (32)$$

By using above formula (4) and simple calculations gives

$$\sin\left(t^\nu \frac{d}{dx}\right) = t^\nu \frac{d}{dx} - t^{3\nu} \cdot \frac{1}{3!} \cdot \frac{d^3}{dx^3} +$$

$$t^{5\nu} \cdot \frac{1}{5!} \cdot \frac{d^5}{dx^5} - \cdots .$$

Now let's calculate the derivatives of $\frac{1}{x}$ and it takes the following form

$$\left(\frac{d}{dx}\right)' \frac{1}{x} = -\frac{1}{x^2}, \quad \left(\frac{d}{dx}\right)'' \frac{1}{x} = \frac{2}{x^3},$$

$$\left(\frac{d}{dx}\right)''' \frac{1}{x} = -\frac{6}{x^4},$$

$$\left(\frac{d}{dx}\right)'''' \frac{1}{x} = \frac{24}{x^5}, \quad \left(\frac{d}{dx}\right)''''' \frac{1}{x} = -\frac{120}{x^6} \cdots .$$

Now let's substituting derivatives of $\frac{1}{x}$ in equation (33) and it takes the following form

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \cdot \frac{1}{x} = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right) \sin\frac{\pi}{2\nu}} \int_0^\infty dt \times$$

$$\frac{1}{x}\left(t^\nu \frac{d}{dx} - t^{3\nu} \cdot \frac{1}{3!} \cdot \frac{d^3}{dx^3} + t^{5\nu} \cdot \frac{1}{5!} \cdot \frac{d^5}{dx^5} - \cdots\right) \qquad (33)$$

and let's do some transformations

$$\frac{1}{x}\left(t^\nu \frac{d}{dx} - t^{3\nu} \cdot \frac{1}{3!} \cdot \frac{d^3}{dx^3} + t^{5\nu} \cdot \frac{1}{5!} \cdot \frac{d^5}{dx^5} - \cdots\right) =$$

$$-t^\nu \cdot \frac{1}{x^2} + t^{3\nu} \cdot \frac{1}{3!} \cdot \frac{3!}{x^4} - t^{5\nu} \cdot \frac{1}{5!} \cdot \frac{5!}{x^6} + \cdots =$$

$$-\frac{1}{x^2} \sum_{n=0}^\infty \frac{(-1)^n \cdot t^{2n\nu+\nu}}{x^{2n}}. \qquad (34)$$

Substituting equation (34) in equation (33) is the following form

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \cdot \frac{1}{x} = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right) \sin\frac{\pi}{2\nu}} \int_0^\infty dt \times$$

$$\left(-\frac{1}{x^2} \sum_{n=0}^\infty \frac{(-1)^n \cdot t^{2n\nu+\nu}}{x^{2n}}\right). \qquad (35)$$

If denote $N(G) = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right) \sin\frac{\pi}{2\nu}}$ and go to the complex plane ξ and to present equation (34) in the integral form:

$$-\frac{1}{x^2} \sum_{n=0}^\infty \frac{(-1)^n \cdot t^{2n\nu+\nu}}{x^{2n}} = \lim_{\epsilon\to 0} \int_\epsilon^\infty dt \cdot t^{2\nu\xi+\nu}$$

and substituting above equation in (33) is the following form

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \cdot \frac{1}{x} = -\frac{N(G)}{x^2} \cdot \frac{1}{2i} \int_{-\beta+i\infty}^{-\beta-i\infty} d\xi \times$$

$$\frac{1}{\sin\pi\xi} \cdot x^{-2\xi} \cdot \lim_{\epsilon\to 0} \int_\epsilon^\infty dt \cdot t^{2\nu\xi+\nu}. \qquad (36)$$

Here

$$\lim_{\epsilon\to 0} \int_\epsilon^\infty dt \cdot t^{2\nu\xi+\nu} = -\lim_{\epsilon\to 0} \frac{\epsilon^2 \nu\xi + \nu + 1}{2\nu\xi + \nu + 1},$$

substituting above equation in (36) is the following form

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \cdot \frac{1}{x} = \frac{N(G)}{x^2} \cdot \frac{1}{2i} \int_{-\beta+i\infty}^{-\beta-i\infty} d\xi \times$$

$$\frac{1}{\sin\pi\xi} \cdot x^{-2\xi} \cdot \lim_{\epsilon\to 0} \frac{t^{2\nu\xi+\nu+1}}{2\nu\xi + \nu + 1}. \qquad (37)$$

Further, we calculate residue at the point

$$\xi = -\frac{(\nu+1)}{2\nu},$$

and here

$$\int_{\beta+i\infty}^{\beta-i\infty} d\xi = 2\pi i.$$

Substituting above equation in (37) is the following form

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \cdot \frac{1}{x} = \frac{N(G)}{x^2} \cdot \frac{1}{2i} 2\pi i \frac{1}{\sin \pi\xi} \cdot x^{-2\xi} \times$$

$$\frac{1}{2\nu} \frac{\left(\xi + \frac{\nu+1}{2\nu}\right)}{\left(\xi + \frac{\nu+1}{2\nu}\right)} = \frac{N(G)}{x^2} \cdot \frac{\pi}{\sin \pi\xi} \cdot x^{-2\xi} \cdot \frac{1}{2\nu}. \quad (38)$$

Lets do some calculations

$$x^{-2\xi} = x^{-2\left(-\frac{(\nu+1)}{2\nu}\right)} = x^{\frac{\nu+1}{\nu}},$$

$$\sin \pi\xi = \sin \pi \cdot \left(-\frac{(\nu+1)}{2\nu}\right) = -\cos \frac{\pi}{2\nu}.$$

Lets back that notaion $N(G) = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right) \sin \frac{\pi}{2\nu}}$ it takes the following form

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \cdot \frac{1}{x} = \frac{1}{\sin \frac{\pi}{2\nu}} \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right)} \times$$

$$\frac{1}{x^2} \cdot \frac{\pi}{-\cos \frac{\pi}{2\nu}} \cdot \frac{1}{2\nu} \cdot x^{\frac{\nu+1}{\nu}}. \quad (39)$$

Here

$$-\left(\sin \frac{\pi}{2\nu} \cdot \cos \frac{\pi}{2\nu}\right) = -\frac{1}{2} \cdot \sin \frac{\pi}{\nu}$$

and after some transformations equation (39) as shown below

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \cdot \frac{1}{x} = \frac{\pi}{-\frac{1}{2} \cdot \sin \frac{\pi}{\nu}} \cdot \frac{1}{\Gamma\left(\frac{1}{\nu}\right)} \cdot \frac{1}{2} \cdot x^{\frac{\nu+1}{\nu}-2}.$$

Here [2]

$$-\sin \frac{\pi}{\nu} \Gamma\left(\frac{1}{\nu}\right) = -\frac{\pi}{\Gamma\left(1 - \frac{1}{\nu}\right)},$$

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \cdot \frac{1}{x} = -\frac{\pi \cdot \Gamma\left(1 - \frac{1}{\nu}\right)}{\pi} \cdot x^{\frac{1-\nu}{\nu}} \quad (40)$$

From equation (32)we obtain a universal formula as show in (41)

$$\boxed{\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \left(\frac{1}{x}\right) = -\Gamma\left(1 - \frac{1}{\nu}\right) \cdot x^{\frac{1-\nu}{\nu}}} \quad (41)$$

here ν -is arbitrary order.

M. $F(x) = \frac{1}{x^2}$

Note that $F(x) = \frac{1}{x^2}$, then using the second general formula (2) and to calculate same as (L) shown as

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \cdot \frac{1}{x^2} = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right) \sin \frac{\pi}{2\nu}} \int_0^\infty dt \times$$

$$\sin\left(t^\nu \frac{d}{dx}\right) \frac{1}{x^2} = \Gamma\left(2 - \frac{1}{\nu}\right) x^{\frac{1}{\nu}-2}. \quad (42)$$

From the equation (42) we obtain a universal formula as show in (43)

$$\boxed{\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \left(\frac{1}{x^2}\right) = \Gamma\left(2 - \frac{1}{\nu}\right) x^{\frac{1}{\nu}-2}} \quad (43)$$

here ν -is arbitrary order.

N. $F(x) = \ln x$

Note that $F(x) = \ln x$, then using the second general formula (2) and to calculate same as (L) shown as

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \cdot \ln x = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right) \sin \frac{\pi}{2\nu}} \int_0^\infty dt \times$$

$$\sin\left(t^\nu \frac{d}{dx}\right) \ln x = \Gamma\left(-\frac{1}{\nu}\right) x^{\frac{1}{\nu}}. \quad (44)$$

From the equation (44) we obtain a universal formula as show in (45)

$$\boxed{\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \ln x = \Gamma\left(-\frac{1}{\nu}\right) x^{\frac{1}{\nu}}} \quad (45)$$

here ν -is arbitrary order.

O. $F(x) = \sqrt{x}$

Note that $F(x) = \sqrt{x}$, then using the second general formula (2) and to calculate same as (L) shown as

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \cdot \sqrt{x} = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right) \sin \frac{\pi}{2\nu}} \int_0^\infty dt \times$$

$$\sin\left(t^\nu \frac{d}{dx}\right) \sqrt{x} = 2^{\frac{1}{\nu}} \cdot x^{\frac{2+\nu}{2\nu}} \left[-\frac{2+2\nu}{\nu} - 1\right]!!. \quad (46)$$

From the equation (46) we obtain a universal formula as show in (47)

$$\boxed{\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \sqrt{x} = 2^{\frac{1}{\nu}} \cdot x^{\frac{2+\nu}{2\nu}} \left[-\frac{2+2\nu}{\nu} - 1\right]!!} \quad (47)$$

here ν -is arbitrary order.

P. $F(x) = \frac{1}{\sqrt{x}}$

Note that $F(x) = \frac{1}{\sqrt{x}}$, then using the second general formula (2) and to calculate same as (L) shown as

$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \cdot \frac{1}{\sqrt{x}} = \frac{\nu}{\Gamma\left(\frac{1}{\nu}\right)\sin\frac{\pi}{2\nu}} \int_0^\infty dt \times$$

$$\sin\left(t^\nu \frac{d}{dx}\right)\frac{1}{\sqrt{x}} = -2^{\frac{1}{\nu}} \cdot x^{\frac{2-\nu}{2\nu}}\left[-\frac{2+2\nu}{\nu}+1\right]!!. \tag{48}$$

From the equation (48) we obtain a universal formula as show in (49)

$$\boxed{\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}} \frac{1}{\sqrt{x}} = -2^{\frac{1}{\nu}} \cdot x^{\frac{2-\nu}{2\nu}}\left[-\frac{2+2\nu}{\nu}+1\right]!!} \tag{49}$$

here ν -is arbitrary order.

IV. APPLICATIONS OF THE NEW FORMULAS

Newton's second law tells us that

$$F = ma = m \cdot \frac{d^2x}{dt^2} = m\ddot{a} = -kx, \tag{50}$$

$$m\frac{d^2x}{dt^2} + kx = 0 \Rightarrow \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{51}$$

Here

$$\omega_0 = \sqrt{\frac{k}{m}}, \tag{52}$$

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0. \tag{53}$$

This is Newton's equation for the harmonic oscillation.

In a previous sections we obtained new formulas for taking fractional derivatives(15, 17, 19, 21, 23, 25, 27, 29, 31, 41, 43, 45, 47, 49). These formulas allow us to construct many different fractional differential equations describing concrete physical processes like the harmonic oscillation, left-and right-moving waves and so on. These particular equivalent equations are:

$$\frac{d^{\rho_1}}{dt^{\rho_1}}a(\omega t) + \omega^2\frac{d^{\rho_2}}{dt^{\rho_2}}a(\omega t) = 0. \tag{54}$$

1. If $\rho_1 = 2, \quad \rho_2 = 0$ we have

$$\frac{d^2}{dt^2}a(\omega t) + \omega^2 a(\omega t) = 0. \tag{55}$$

Here $a(\omega t) = \sin(\omega t)$

$$\frac{d^2}{dt^2}\sin(\omega t) = -\omega^2 \sin\omega t,$$

$$-\omega^2\sin\omega t + \omega^2 \sin\omega t = 0.$$

This equation showed to materialise when integer derivatives give harmonic oscillator solution like

$$a_1(\omega t) = A\sin(\omega t + \varphi),$$

$$a_2(\omega t) = A\cos(\omega t + \varphi).$$

Now we consider fractional derivatives.

2.

$$\left(\frac{d}{dt}\right)^{\frac{1}{2}} c_i(\omega t) + \omega\left(\frac{d}{dt}\right)^{-\frac{1}{2}} c_i(\omega t) = \sqrt{2\omega}c_i(\omega t) \tag{56}$$

Here $c_i(\omega t) = \sin(\omega t)$.

Firstly let's calculate derivatives of $\sin(\omega t)^{\frac{1}{2}}$ and $\sin(\omega t)^{-\frac{1}{2}}$ functions were the following forms

$$\left(\frac{d}{dt}\right)^{\frac{1}{2}}\sin(\omega t) = \omega^{\frac{1}{2}} \cdot \sin\left(\omega t + \frac{\pi}{4}\right) =$$

$$\sqrt{\omega}\left(\sin\omega t \cdot \cos\frac{\pi}{4} + \cos\omega t \cdot \sin\frac{\pi}{4}\right).$$

$$\left(\frac{d}{dt}\right)^{\frac{1}{2}}\sin(\omega t) = \frac{\sqrt{2\omega}}{2}(\sin\omega t + \cos\omega t)$$

and

$$\left(\frac{d}{dt}\right)^{-\frac{1}{2}}\sin(\omega t) = \omega^{-\frac{1}{2}} \cdot \sin\left(\omega t - \frac{\pi}{4}\right) =$$

$$\frac{1}{\sqrt{\omega}}\left(\sin\omega t \cdot \cos\frac{\pi}{4} - \cos\omega t \cdot \sin\frac{\pi}{4}\right).$$

$$\left(\frac{d}{dt}\right)^{-\frac{1}{2}}\sin(\omega t) = \frac{\sqrt{2}}{2\sqrt{\omega}}(\sin\omega t - \cos\omega t).$$

Substituting above equations in (56) is the following form

$$\frac{\sqrt{2\omega}}{2}(\sin\omega t + \cos\omega t) +$$

$$\omega\left(\frac{\sqrt{2}}{2\sqrt{\omega}}(\sin\omega t - \cos\omega t)\right) = \sin\omega t\sqrt{2\omega}. \tag{57}$$

Here $\sqrt{2\omega}$ - coefficient depends on derivatives order.

Now, let's check fractional derivatives of $\sin\omega t$ satisfying equation (54)

$$\frac{\sqrt{2\omega}}{2}(\sin\omega t + \cos\omega t) +$$

$$\omega\left(\frac{\sqrt{2}}{2\sqrt{\omega}}\left(\sin\omega t-\cos\omega t\right)\right)=\sqrt{2\omega}\sin\omega t. \tag{58}$$

Let's do elementary calculations

$$\frac{\sqrt{2\omega}\sin\omega t+\sqrt{2\omega}\cos\omega t}{2}+$$

$$\frac{\omega\sqrt{2}\sin\omega t-\omega\sqrt{2}\cos\omega t}{2\sqrt{\omega}}=\sqrt{2\omega}\sin\omega t.$$

After some elementary calculations as shown as

$$2\omega\sqrt{2}\sin\omega t-2\omega\sqrt{2}\sin\omega t=0.$$

It showed us when calculate fractional derivatives it satisfying Newton's equations, harmonic oscillator solution like

$$c_1(\omega t)=A\sin(\omega t+\varphi),$$

$$c_2(\omega t)=A\cos(\omega t+\varphi).$$

Now we consider other fractional derivatives .

3.

$$\left(\frac{d}{dt}\right)^{\frac{3}{2}}d_i(\omega t)+\omega^3\left(\frac{d}{dt}\right)^{-\frac{3}{2}}d_i(\omega t)=$$

$$-\omega\sqrt{2\omega}d_i(\omega t) \tag{59}$$

let's repeat the calculations

$$d_1(\omega t)=A\sin(\omega t+\varphi),$$

$$d_1(\omega t)=A\cos(\omega t+\varphi).$$

Now we consider other fractional derivatives.

4.

$$\left(\frac{d}{dt}\right)^{\frac{1}{4}}e_i(\omega t)+\sqrt{\omega}\left(\frac{d}{dt}\right)^{-\frac{1}{4}}e_i(\omega t)=$$

$$2\omega^{\frac{1}{4}}\cos\frac{\pi}{8}e_i(\omega t) \tag{60}$$

let's repeat the calculations

$$e_1(\omega t)=A\sin(\omega t+\varphi),$$

$$e_1(\omega t)=A\cos(\omega t+\varphi).$$

Now we consider other fractional derivatives.

5.

$$\sqrt{2\omega}\left(\frac{d}{dt}\right)^{-\frac{1}{2}}k_i(\omega t)-\omega\left(\frac{d}{dt}\right)^{-1}k_i(\omega t)=k_i(\omega t) \tag{61}$$

let's repeat the calculations

$$k_1(\omega t)=A\sin(\omega t+\varphi),$$

$$k_1(\omega t)=A\cos(\omega t+\varphi).$$

We see that equations (54)-(61) describe the harmonic oscillation process. In the usual traditional case this oscillation satisfies Newton's equation

$$m\frac{d^2x(t)}{dt^2}=-kx(t)\Rightarrow\ddot{x}(t)+\omega^2x(t)=0$$

solution of which is

$$x(t)=A\sin(\omega t+\varphi).$$

For left- and right- moving waves $f(x-vt)$ and $f(x+vt)$ one can change notation in previous formulas:

$$t\to x,\quad \omega\to 1,\quad \omega t\to x-vt\quad or\quad x+vt,$$

$$a(\omega t)\Rightarrow f(x-vt),\quad f(x+vt)$$
$$\text{or}\quad F=A_1f(x-vt)+A_2f(x+vt),$$

Then all equations (54)-(61) have solutions

$$A_i\sin(x-vt)\quad\text{and}\quad A_j\cos(x-vt)$$
$$\text{or}\quad A_k\sin(x+vt)\quad\text{and}\quad A_n\cos(x+vt)$$

and their linear combinations like

$$A_i\sin(x-vt)+A_k\sin(x+vt)$$

or

$$A_j\cos(x-vt)+A_n\cos(x+vt)$$

so on.

V. RESULTS

Finally we found universal formulas for taking fractional derivatives.

1.
$$\left(\frac{d}{dx}\right)^{\rho}\sin x=\sin\left(x+\frac{\pi}{2}\rho\right)$$
here ρ -is any arbitrary number.

2.
$$\left(\frac{d}{dx}\right)^{\rho}\cos x=\cos\left(x+\frac{\pi}{2}\rho\right)$$

3.
$$\left(\frac{d}{dx}\right)^{\rho}\sin ax=a^{\rho}\sin\left(ax+\frac{\pi}{2}\rho\right)$$

4.
$$\left(\frac{d}{dx}\right)^{\rho}\cos ax=a^{\rho}\cos\left(ax+\frac{\pi}{2}\rho\right)$$

5.
$$\left(\frac{d}{dx}\right)^{\rho}e^{ax}=a^{\rho}e^{ax}$$

6.
$$\left(\frac{d}{dx}\right)^{\rho}a^{x}=(\ln a)^{\rho}a^{x},\quad a>1$$

7.
$$\left(\frac{d}{dx}\right)^{\rho}e^{-ax}=-a^{\rho}e^{-ax}$$

8.
$$\left(\frac{d}{dx}\right)^{\rho}\mathbf{sh}ax=a^{\rho}\mathbf{ch}ax$$

9.
$$\left(\frac{d}{dx}\right)^{\rho}\mathbf{ch}ax=a^{\rho}\mathbf{sh}ax$$

10.
$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}}\left(\frac{1}{x}\right)=-\Gamma\left(1-\frac{1}{\nu}\right)\cdot x^{\frac{1-\nu}{\nu}}$$
here ν -is any arbitrary number.

11.
$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}}\ln x=\Gamma\left(-\frac{1}{\nu}\right)x^{\frac{1}{\nu}}$$

12.
$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}}\left(\frac{1}{x^2}\right)=\Gamma\left(2-\frac{1}{\nu}\right)x^{\frac{1}{\nu}-2}$$

13.
$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}}\sqrt{x}=2^{\frac{1}{\nu}}\cdot x^{\frac{2+\nu}{2\nu}}\left[-\frac{2+2\nu}{\nu}-1\right]!!$$

14.
$$\left(\frac{d}{dx}\right)^{-\frac{1}{\nu}}\frac{1}{\sqrt{x}}=-2^{\frac{1}{\nu}}\cdot x^{\frac{2-\nu}{2\nu}}\left[-\frac{2+2\nu}{\nu}+1\right]!!$$

VI. CONCLUSION

We studied Newton's equations for the harmonic oscillations using newly generated formula derived in our study and we conclude that Newton's equations are not limited to integers, but can be well presented using fractional derivatives.

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