Square-Root Differential Equations for Dark Matter Particles

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We propose the square root operator formalism for description of motion for a dark particle candidate. It turns out that such type particle moves randomly and fills in whole space. It seems that such particles do not create material structure of universe, like atoms, molecules, stars, planets and etc. due to their stochastic motion.

I. INTRODUCTION

Recently, concept of dark particles and dark energy plays a vital role in the cosmological theory. In last decates many experimental and theoretical studies are carried out in this direction and have obtained many interesting results. However, nature of dark particles and dark energy is unclear and does not understood very well.

Dark matter is a hypothetical type of matter distinct from baryonic matter (ordinary matter such as protons and neutrons), electrons, neutrinos and dark energy. Dark matter has never been directly observed; however, its existence would explain a number of otherwise puzzling astronomical observations [1], and its properties are inferred from its gravitational effects such as the motions of baryonic matter [2], gravitational lensing, its influence on the universe's large-scale structure, on the formation of galaxies, and its effect on the cosmic microwave background (CMB).

The standard model of cosmology indicates that the total mass- energy of the universe contains 4.9% ordinary matter, 26,8% dark matter and 68,3% dark energy [3]. Thus, dark matter constitutes 84,5% of total mass, while dark energy plus dark matter constitute 95,1% of total mass-energy content [4].

The dark matter hypothesis plays a central role in current modeling of cosmic structure formation, galaxy formation and evolution, and on explanations of the anisotropies observed in the cosmic microwave background (CMB).

The most widely accepted hypothesis on the form for dark matter is that its composed of **weakly interacting** massive particles that interact only through **gravity** and the **weak force**.

In this article, we propose the **square-root operator formalism** for the description of a dark matter particle which possesses random properties with random momentum or random mass with the definite probabilistic measure:

$$\omega(\rho) = \frac{1}{\pi}\frac{1}{\sqrt{1-\rho^2}} \tag{1}$$

with properties:

$$\int_{-1}^{1} d\rho\omega(\rho) = 1, \tag{2}$$

$$\int_{-1}^{1} d\rho\rho\omega(\rho) = 0, \tag{3}$$

$$\int_{-1}^{1} d\rho\rho^2\omega(\rho) = \frac{1}{2}. \tag{4}$$

II. THE GREEN FUNCTIONS OF THE USUAL PARTICLES

We know that ordinary particles such as scalar, spinor, vector fields obey the usual differential equations [5,6]

A. Scalar particle

The homogeneous Klein-Gordon equation

$$(m^2 - \Box)\mathcal{D}(x) = 0 \tag{5}$$

has the solution

$$\mathcal{D}(x) = \frac{1}{2\pi}\epsilon(x^0)\delta(\lambda) - \frac{m}{4\pi\sqrt{\lambda}}\epsilon(x^0)\theta(\lambda)J_1(m\sqrt{\lambda}), \tag{6}$$

where $\lambda = x_0^2 - \overrightarrow{x}^2, \qquad \Box = \triangle - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$.

Moreover, the fundamental solution (the causal Green function) of the Klein-Gordon-Fock operator

$$(m^2 - \Box)G^c(x) = \delta^4(x) \tag{7}$$

is given by

$$G^c(x) = \frac{1}{(2\pi)^4}\int d^4k\frac{e^{-ikx}}{m^2-k^2-i\epsilon}. \tag{8}$$

An explicit form of this function is

$$\begin{aligned} G^c(x) &= \frac{1}{4\pi}\delta(\lambda) - \frac{m}{8\pi\sqrt{\lambda}}\theta(\lambda)[J_1(m\sqrt{\lambda}) - iN_1(m\sqrt{\lambda})] \\ &+ \frac{mi}{4\pi^2\sqrt{-\lambda}}\theta(-\lambda)K_1(m\sqrt{-\lambda}). \end{aligned} \tag{9}$$

Here $J_1(x), N_1(x)$ and $K_1(x)$ are the Bessel, the Bessel function of the second kind or the Neumann one (also denoted by $Y_1(x)$) and the modified Bessel function of the second kind (sometimes $K_1(x)$ is called the Mac' Donald function), respectively.

B. An even solution of the inhomogeneous D'Alembert equation

$$\Box G^0(x) = -\delta^4(x) \tag{10}$$

is

$$G^0(x) = G^c(x)|_{m=0} = \frac{1}{4\pi}(\delta(\lambda) - \frac{i}{\pi\lambda}) = \frac{1}{4\pi}\delta_+(-\lambda) \equiv \frac{1}{4\pi^2}\int_0^\infty dx e^{ix(-\lambda)}. \tag{11}$$

In this case, the photon Green function can be represented in the well-known form

$$G^{ph}_{\mu\nu}(x) = \frac{g_{\mu\nu}}{(2\pi)^4 i}\int d^4k \frac{e^{-ikx}}{k^2 + i\epsilon} \tag{12}$$

C. The fundamental solution of the Green function for vector field

satisfies the Prock equation

$$\left(g^{\mu\nu} + \frac{1}{m^2}\frac{\partial^2}{\partial x^\mu \partial x^\nu}\right) G^c_{\mu\nu}(x) = \delta^4(x), \tag{13}$$

where

$$G^c_{\mu\nu}(x) = \left(g^{\mu\nu} + \frac{1}{m^2}\frac{\partial^2}{\partial x^\mu \partial x^\nu}\right) G^c(x) \tag{14}$$

or

$$G^c_{\mu\nu}(x) = \frac{1}{(2\pi)^4}\int d^4k e^{-ikx}(g^{\mu\nu} - \frac{k_\mu k_\nu}{m^2}) \times \frac{1}{m^2 - k^2 - i\epsilon}. \tag{15}$$

Here $G^c(x)$ is the Green function (9) for the Klein-Gordon equation (7). In this case, homogeneous equation of (13) has the solution

$$\mathcal{D}_{\mu\nu}(x) = (g^{\mu\nu} + \frac{1}{m^2}\frac{\partial^2}{\partial x_\mu \partial x_\nu})\mathcal{D}(x), \tag{16}$$

where $\mathcal{D}(x)$ is given by the formula (6).

D. The fundamental solution or the Green function for the Dirac equation

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} + m\right) S^c(x) = \delta^4(x) \tag{17}$$

is given by the formulas

$$S^c(x) = \frac{1}{(2\pi)^4}\int d^4p e^{-ipx}\frac{m + \hat{p}}{m^2 - p^2 - i\epsilon}, \tag{18}$$

where $\hat{p} = p_0\gamma^0 - \vec{p}\vec{\gamma}$, γ^μ are the Dirac γ-matrices.

III. THE FUNDAMENTAL SOLUTION OF THE GREEN FUNCTION FOR THE WEYL EQUATION

$$\sqrt{m^2 - \Box} W^c(x) = \delta^4(x). \tag{19}$$

Long time ago H.Weyl [7] proposed the square-root operator formalism, like (19) in the field theory. However, because of mathematical difficulty how to work with the differential operator under the square-root, this formalism did not developed and instead of which Klein-Gordon formalism was accepted.

It turns out that the Weyl equation (19) has remarkable properties and gives stochastic solutions over momentum variables $p_\mu \Rightarrow \rho p_\mu$ or equivalently over mass value $m \Rightarrow \rho m$, where ρ is random variable with the measure (1). To get these properties, we consider the Green function $\tilde{W}^c(p)$ in the momentum space [8]

$$\tilde{W}^c(p) = \frac{1}{\sqrt{m^2 - p^2}} \tag{20}$$

and use the Feynman parametrization formula

$$\frac{1}{a^{n_1}b^{n_2}} = \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)}\int_0^1 dx x^{n_1-1}(1-x)^{n_2-1} \times \frac{1}{[ax + b(1-x)]^{n_1+n_2}} \tag{21}$$

In our case $n_1 = n_2 = 1/2$, $\Gamma(1/2) = \sqrt{\pi}$, $m^2 - p^2 = (m - \hat{p})(m + \hat{p})$, $\hat{p} = \gamma_\mu p^\mu$. The result reads

$$\tilde{W}^c(p) = \frac{1}{\pi}\int_{-1}^1 d\rho \frac{1}{\sqrt{1-\rho^2}}\frac{m + \hat{p}\rho}{m^2 - p^2\rho^2}. \tag{22}$$

An another equivalent representation (20)

$$\tilde{W_1}^c(p) = \frac{-1}{i\sqrt{p^2 - m^2}}$$

gives stochasicity over mass value m:

$$\tilde{W_1}^c(p) = \frac{1}{\pi}\int_{-1}^1 d\rho \frac{1}{\sqrt{1-\rho^2}}\frac{1}{i}\frac{m\rho + \hat{p}}{m^2\rho^2 - p^2}. \tag{23}$$

These two representations (22) and (23) are absolutely equivalent and only have different physical interpretation. The first case (22) gives the well-known spinor propagator or the causal Green function

$$W^c_{sp}(x) = \frac{1}{\pi}\int_{-1}^1 \frac{d\rho}{\sqrt{1-\rho^2}} \times \frac{1}{(2\pi)^4}\int d^4p e^{-ipx}\frac{m + \hat{p}\rho}{m^2 - p^2\rho^2 - i\epsilon} \tag{24}$$

with stochastic momentum $p^\mu \to \rho p^\mu$ and stochastic energy value

$$\omega_1 = \frac{1}{|\rho|}\sqrt{m^2 + \vec{p}^{\,2}\rho^2}. \quad (25)$$

Moreover, due to property

$$\frac{1}{\pi}\int_{-1}^{1} d\rho \frac{\rho}{\sqrt{1-\rho^2}} = 0,$$

the case (22) includes also scalar particles with stochastic momentum and its Green function takes the form

$$W^c_{sc}(x) = \frac{m}{\pi}\int_{-1}^{1} d\rho \frac{1}{\sqrt{1-\rho^2}}\frac{1}{(2\pi)^4} \times \int d^4p e^{-ipx}\frac{1}{m^2 - p^2\rho^2 - i\epsilon}. \quad (26)$$

The case (23) leads to the spinor propagator with the random mass $m \to m\rho$:

$$W^c_{1sp}(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{d\rho}{\sqrt{1-\rho^2}}\frac{1}{(2\pi)^4 i} \times \int d^4p e^{-ipx}\frac{m\rho + \hat{p}}{m^2\rho^2 - p^2 - i\epsilon} \quad (27)$$

with stochastic energy

$$\omega_2 = \sqrt{m^2\rho^2 + \vec{p}^{\,2}}. \quad (28)$$

Averaged energy (25) for the first case gives singular value:

$$< \omega_1 >= \frac{2}{\pi}\int_{0}^{1} \frac{d\rho}{\rho}\frac{\sqrt{m^2 + \vec{p}^{\,2}\rho^2}}{\sqrt{1-\rho^2}}. \quad (29)$$

In contrary, the second case (23) gives a finite value for an averaged energy

$$< \omega_2 > = \frac{2}{\pi}\int_{0}^{1} d\rho \frac{1}{\sqrt{1-\rho^2}}\sqrt{m^2\rho^2 + \vec{p}^{\,2}} = \frac{2\sqrt{m^2 + \vec{p}^{\,2}}}{\pi} E\left(\frac{\pi}{2}, \frac{m}{\sqrt{m^2 + \vec{p}^{\,2}}}\right) \quad (30)$$

and

$$< \omega_2^2 >= \frac{1}{2}m^2 + \vec{p}^{\,2}. \quad (31)$$

Here $E\left(\frac{\pi}{2}, \frac{m}{\sqrt{m^2+\vec{p}^{\,2}}}\right)$ is the elliptic integral of the second kind

$$E(\varphi, k) = \int_0^\varphi \sqrt{1 - k^2\sin^2\alpha}\, d\alpha.$$

Notice that the first case leads to interesting consequences that in the square-root formalism averaged rest mass goes to infinite

$$< \omega_1 > |_{\vec{p}=0} = \infty.$$

It means that the first case gives super heavy dark particle. It is natural that in this formalism the Einstein formula $E = mc^2$ for usual matter particles does changed and acquires the form

$$< E_0 >= \frac{2m}{\pi}c^2 \quad (32)$$

due to formula (30).

In this paper, we propose that square-root or Weyl particles maybe played a role as dark matter particles in the whole Universe. Due to random diffusion, like the Brown motion detection of dark matter particles by experiments encountries difficults, therefore they fill in whole empty space and do not make up usual matter structure.

IV. EXPLICIT FORMS OF GREEN FUNCTIONS FOR SQUARE-ROOT PARTICLES (DARK PARTICLES) IN X-SPACE

A. Scalar particle case with random momentum $p_\mu\rho$

In accountancy with the formulas (9) and (26), one gets

$$W^c_{sc}(x) = \frac{m}{\pi}\int_{-1}^{1} d\rho \frac{1}{\sqrt{1-\rho^2}}\frac{1}{\rho^2} \times \left\{\frac{1}{4\pi}\delta(\lambda) - \frac{m}{8\pi\sqrt{\lambda}\rho}\theta(\lambda)\left[J_1\left(\frac{m}{\rho}\sqrt{\lambda}\right) - iN_1\left(\frac{m}{\rho}\sqrt{\lambda}\right)\right] + \frac{mi}{4\pi^2\sqrt{-\lambda}}\frac{1}{\rho}\theta(-\lambda)K_1\left(\frac{m}{\rho}\sqrt{-\lambda}\right)\right\}. \quad (33)$$

B. Spinor particle with stochastic momentum $p_\mu\rho$

By using the formulas (24) and (9) one gets

$$W^c_{sp}(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{d\rho}{\sqrt{1-\rho^2}}\frac{1}{\rho}\left(i\gamma^\nu\frac{\partial}{\partial x^\nu} + \frac{m}{\rho}\right) \left\{\frac{1}{4\pi}\delta(\lambda) - \frac{m}{8\pi\rho\sqrt{\lambda}}\theta(\lambda)\times \times \left[J_1\left(\frac{m}{\rho}\sqrt{\lambda}\right) - iN_1\left(\frac{m}{\rho}\sqrt{\lambda}\right)\right]\right\}. + \frac{im}{4\pi^2\rho\sqrt{-\lambda}}\theta(-\lambda)K_1\left(\frac{m}{\rho}\sqrt{-\lambda}\right) \quad (34)$$

C. Spinor particle with stochastic particle's mass $m\rho$

Taking into account the formulas (9) and (27) it easy to find the following causal Green-Function for a dark particle

$$\begin{aligned} W^c_{sp}(x) &= \frac{1}{\pi}\int_{-1}^{1} d\rho \frac{1}{\sqrt{1-\rho^2}}\left(i\gamma^\nu \frac{\partial}{\partial x^\nu} + m\rho\right) \\ &\left\{\frac{1}{4\pi}\delta(\lambda) - \frac{m\rho}{8\pi\sqrt{-\lambda}}\theta(\lambda)\times\right. \\ &\left[J_1\left(m\rho\sqrt{\lambda}\right) - iN_1\left(m\rho\sqrt{\lambda}\right)\right] \\ &+ \left.\frac{im\rho}{4\pi^2\sqrt{-\lambda}}\theta(-\lambda)K_1\left(m\rho\sqrt{-\lambda}\right)\right\}. \quad (35) \end{aligned}$$

From the formulas (33)-(35) one can conclude that the Green function or propagator of a dark scalar particle (33) and causal Green function for a dark spinor particle (34) are singular functions with respect to integration over random variable ρ and therefore they can not play a role as causal Green functions.

In contrary, the causal Green function (35) is finite. It means that stochasticity in mass variable for a square-root or dark particle has definite physical meaning.

In conclusion, notice that from the formulas (23) and (27) one can conclude that a dark neutrino like particle coincides with the usual neutrino in the limit $m \to 0$, due to the formula (2). It mean that if an usual neutrino possesses even extremely small mass then neutrinos play a vital role in dark matter content of the Universe.

V. A SOLUTION OF THE SQUARE-ROOT KLEIN-GORDON EQUATION

Let us consider the equation

$$\sqrt{(m^2-\Box)}\Phi(x) = 0. \quad (36)$$

Here we use the following formal transformation

$$\frac{\sqrt{(m^2-\Box)}\cdot\sqrt{(m^2-\Box)}}{\sqrt{(m^2-\Box)}}\Phi(x) \quad (37)$$

$$= \frac{1}{\pi}\int_{-1}^{1}\frac{d\rho}{\sqrt{1-\rho^2}}\frac{m\rho+\hat{\partial}}{m^2\rho^2-\Box}(m^2-\Box)\Phi(x) = 0.$$

In the momentum representation a solution of equation (37) takes the form

$$\begin{aligned} \Phi(x) &= \frac{1}{\pi}\int_{-1}^{1}\frac{d\rho}{\sqrt{1-\rho^2}}\frac{1}{(2\pi)^4}\int d^4p e^{ipx}\tilde{\varphi}(p) \\ &\times\ \delta(m^2-p^2)\frac{m\rho+\hat{p}}{m^2\rho^2-p^2}, \quad (38) \end{aligned}$$

where

$$\Phi(x) = \frac{1}{(2\pi)^4}\int d^4p e^{ipx}\tilde{\varphi}(p),$$

$$\delta(m^2-p^2) = \frac{1}{2\omega_p}\left[\delta(p_0+\omega_p)+\delta(p_0-\omega_p)\right],$$

$$\omega_p = \sqrt{m^2+\vec{p}^{\,2}}.$$

An another equivalent representation for (37) takes the form

$$\frac{1}{\pi}\int_{-1}^{1}\frac{d\rho}{\sqrt{1-\rho^2}}\frac{m+\rho\hat{\partial}}{m^2-\rho^2\Box}(m^2-\Box)\Phi(x) = 0 \quad (39)$$

or

$$\begin{aligned} \Phi(x) &= \frac{1}{\pi}\int_{-1}^{1}\frac{d\rho}{\sqrt{1-\rho^2}}\times\frac{1}{(2\pi)^4}\int d^4p e^{ipx}\delta(m^2 \\ &-\ p^2)\frac{m+\rho\hat{p}}{m^2-\rho^2p^2}\tilde{\varphi}(p). \quad (40) \end{aligned}$$

Here

$$\frac{1}{m^2-\rho^2p^2} = \frac{1}{m^2(1+\rho)^2+2\rho^2\vec{p}^{\,2}}, \quad \text{for} \quad p_0 = -\omega_p,$$

or

$$\frac{1}{m^2-\rho^2p^2} = \frac{1}{m^2(1-\rho)}, \quad \text{for} \quad p_0 = \omega_p$$

$$\hat{p} = \gamma^\nu p_\nu, \quad \hat{\partial} = i\gamma^\nu\frac{\partial}{\partial x^\nu}.$$

VI. THE PAULI-JORDAN SOLUTION OF THE SQUARE-ROOT KLEIN-GORDON EQUATION

In this case, equation (36) takes the form

$$\sqrt{m^2-\Box}P(x) = 0 \quad (41)$$

where

$$\begin{aligned} P_1(x) &= \frac{1}{\pi}\int_{-1}^{1}\frac{d\rho}{\sqrt{1-\rho^2}}\int_0^\infty d\alpha e^{-\alpha(m^2\rho^2-\Box)} \\ &\times\ \left(m\rho+i\gamma^\nu\frac{\partial}{\partial x^\nu}\right)\mathcal{D}_{GF}(x). \quad (42) \end{aligned}$$

or

$$\begin{aligned} P_2(x) &= \frac{1}{\pi}\int_{-1}^{1}\frac{d\rho}{\sqrt{1-\rho^2}}\int_0^\infty d\alpha e^{-\alpha(m^2-\rho^2\Box)}. \\ &\cdot\left(m+i\rho\gamma^\nu\frac{\partial}{\partial x^\gamma}\right)\mathcal{D}_{GF}(x). \end{aligned}$$

Here $\mathcal{D}_{GF}(x)$ is given by the formula (6).

VII. DARK PHOTONS

An Even Solution of the Square-Root Inhomogeneous D'Alembert equation

$$\sqrt{\Box}W_0(x) = \delta^4(x) \tag{43}$$

is given by the following two equivalent expressions in the momentum space:

$$\tilde{W}_0^1(p) = \frac{1}{\pi}\int_{-1}^{1} d\rho \frac{1}{\sqrt{1-\rho^2}} \frac{p_0 + \hat{\vec{p}}\rho}{p_0^2 - \vec{p}^2\rho^2} \tag{44}$$

or

$$\tilde{W^2}_0(p) = \frac{1}{\pi}\int_{-1}^{1} d\rho \frac{1}{\sqrt{1-\rho^2}} \frac{\rho p_0 + \hat{\vec{p}}}{p_0^2\rho^2 - \vec{p}^2}, \tag{45}$$

where $\hat{\vec{p}} = \sigma^i p_i$, σ^i are the Pauli matrices.

These two formulas (44) and (45) in x-space take the forms

$$W_0^1(x) = \frac{1}{\pi}\int_{-1}^{1} d\rho \frac{1}{\sqrt{1-\rho^2}} \left(i\rho^{-1}\frac{\partial}{\partial t} + i\vec{\sigma}\frac{\partial}{\partial \vec{x}} \right) \cdot \cdot \frac{1}{4\pi}\left[\delta(\lambda') - \frac{i}{\pi\lambda'}\right], \tag{46}$$

$$\lambda' = x_0^2\rho^2 - \vec{x}^2,$$

and

$$W_0^2(x) = \frac{1}{\pi}\int_{-1}^{1} d\rho \frac{1}{\rho\sqrt{1-\rho^2}} \left(i\rho\frac{\partial}{\partial t} + i\vec{\sigma}\frac{\partial}{\partial \vec{x}} \right) \cdot \cdot \frac{1}{4\pi}\left[\delta(\lambda'') - \frac{i}{\pi\lambda''}\right], \tag{47}$$

$$\lambda'' = x_0^2/\rho^2 - \vec{x}^2.$$

Here

$$G_0(x) = \frac{1}{4\pi}\left[\delta(\lambda) - \frac{i}{\pi\lambda}\right] \tag{48}$$

corresponds to the Green function (the causal function) for a massless particle, like the photon, which can be represented in the well-known form

$$G_0^{ph}(x) = \frac{g_{\mu\nu}}{(2\pi)^4}\frac{1}{i}\int d^4p \frac{e^{-ipx}}{p^2 + i\epsilon} \tag{49}$$

Averaged energy for a dark photon for the second case takes the form

$$< E_{20}^{\gamma} >= \frac{1}{\pi}\int_{-1}^{1} d\rho \frac{1}{\sqrt{1-\rho^2}}\frac{1}{|\rho|}|\vec{p}| = \frac{2}{\pi}\int_{0}^{1} \frac{d\rho}{\rho\sqrt{1-\rho^2}}|\vec{p}|. \tag{50}$$

This integral is diverged and therefore the second case does not accepted. Moreover, the first case leads to the finite energy form

$$< E_{10}^{\gamma} > = \frac{1}{\pi}\int_{-1}^{1} d\rho \frac{1}{\sqrt{1-\rho^2}}|\rho||\vec{p}| = \frac{2}{\pi}\int_{0}^{1} d\rho\rho \frac{1}{\sqrt{1-\rho^2}}|\vec{p}| = \frac{2}{\pi}|\vec{p}| \tag{51}$$

for the dark photon.

Thus, family of dark matter particles consist of dark scalar, dark spinors, dark photons and dark neutrinos which coincide with usual neutrinos. All these particles possess stochastic properties with respect to the probabilistic measure (1).

Appendix A

It turns out that the probabilistic measure (1) plays a role as a filter or an intermediate mathematical trick in the square-root differential calculus. Due to this filter solutions of square-root differential equations describe wave properties of dark matter particles. To show this we consider very simple system-harmonic motion defining by the differential equation

$$\left(a^2 + \frac{d^2}{dt^2}\right)x(t) = 0, \tag{52}$$

solution of which is

$$x(t) = A\sin at. \tag{53}$$

Now let us turn to the square-root equation

$$\sqrt{a^2 + \frac{d^2}{dt^2}}X(t) = 0, \tag{54}$$

where the probabilistic measure (1) appears:

$$\frac{1}{\sqrt{a^2 + \frac{d^2}{dt^2}}}f(t) = \frac{1}{\pi}\int_{-1}^{1}\frac{d\rho}{\sqrt{1-\rho^2}}\int_{0}^{\infty} d\alpha e^{-\alpha(a^2+\rho^2\frac{d^2}{dt^2})} \cdot \cdot \left(a + i\rho\frac{d}{dt}\right)f(t). \tag{55}$$

Here

$$e^{-\alpha\rho^2\frac{d^2}{dt^2}} = 1 - \alpha\rho^2\frac{d^2}{dt^2} + \frac{\alpha^2\rho^4}{2!}\frac{d^4}{dt^4} - \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\alpha^n\rho^{2n}\frac{d^{2n}}{dt^{2n}},$$

and we use the following calculations

1

$$\frac{1}{\pi}\int_{-1}^{1}\frac{d\rho}{\sqrt{1-\rho^2}}\rho^{2n}=\frac{1}{\sqrt{\pi}}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}.$$

2.

$$\int_{0}^{\infty}d\alpha\alpha^n e^{-\alpha a^2}=(a^2)^{-1-n}\Gamma(n+1).$$

3.

$$\frac{1}{\pi}\int_{-1}^{1}\frac{d\rho}{\sqrt{1-\rho^2}}\rho^{2n+1}=0.$$

Then, we have nice formula

$$\hat{\mathcal{D}}f(x) = \frac{1}{\sqrt{a^2+\frac{d^2}{dt^2}}}f(t)=\frac{1}{a\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n \cdot \frac{\Gamma(n+1/2)}{n!}\frac{1}{a^{2n}}\left(\frac{d^2}{dt^2}\right)^n f(t). \quad (56)$$

In particular:

$$\hat{\mathcal{D}}C=\frac{1}{a}C, \qquad \hat{\mathcal{D}}t=\frac{1}{a}t,$$

$$\hat{\mathcal{D}}t^2=\frac{1}{a}t^2-\frac{1}{a^3},$$

$$\begin{aligned}
&\hat{\mathcal{D}}\sin bt=\Lambda(a,b)\sin bt,\\
&\hat{\mathcal{D}}\cos bt=\Lambda(a,b)\cos bt,\\
&\hat{\mathcal{D}}e^{ibt}=\Lambda(a,b)e^{ibt},\\
&\hat{\mathcal{D}}e^{-ibt}=\Lambda(a,b)e^{-ibt},\\
&\hat{\mathcal{D}}e^{bt}=\Lambda'(a,b)e^{bt},\\
&\hat{\mathcal{D}}e^{-bt}=\Lambda'(a,b)e^{-bt},
\end{aligned}$$

and so on. Here

$$\begin{aligned}
\Lambda(a,b) &= \frac{1}{a\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{n!}\left(\frac{b^2}{a^2}\right)^n\\
&= \frac{1}{a}\left(1-\frac{b^2}{a^2}\right)^{-1/2}, \qquad (57)
\end{aligned}$$

$$\begin{aligned}
\Lambda'(a,b) &= \frac{1}{a\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+1/2)}{n!}\left(\frac{b^2}{a^2}\right)^n\\
&= \frac{1}{a}\left(1+\frac{b^2}{a^2}\right)^{-1/2}. \qquad (58)
\end{aligned}$$

So that

$$\hat{\mathcal{D}}\sin bt=\frac{\sin bt}{\sqrt{a^2-b^2}},$$
$$\hat{\mathcal{D}}\cos bt=\frac{\cos bt}{\sqrt{a^2-b^2}}.$$

Finally, equation (54) takes the form

$$\begin{aligned}
\hat{N}X(t) &= \sqrt{a^2+\frac{d^2}{dt^2}}X(t)=\frac{\left(a^2+\frac{d^2}{dt^2}\right)}{\sqrt{a^2+\frac{d^2}{dt^2}}}X(t) \qquad (59)\\
&= \left(a^2+\frac{d^2}{dt^2}\right)\hat{\mathcal{D}}X(t)=\hat{\mathcal{D}}\left(a^2+\frac{d^2}{dt^2}\right)X(t).
\end{aligned}$$

In particular,

$$\sqrt{a^2+\frac{d^2}{dt^2}}\sin bt=\frac{a^2-b^2}{\sqrt{a^2-b^2}}\sin bt,$$
$$\sqrt{a^2+\frac{d^2}{dt^2}}\cos bt=\frac{a^2-b^2}{\sqrt{a^2-b^2}}\cos bt.$$

Therefore, the square-root differential equation

$$\sqrt{a^2+\frac{d^2}{dt^2}}X(t)=0 \qquad (60)$$

describes also harmonic oscillator $X(t) = A\sin at$ due to filter properties of the probabilistic measure (1). Generalization of the equation (60)

$$\sqrt{m^2-\Box}G_c(x)=\delta^4(x) \qquad (61)$$

or

$$\frac{(m^2-\Box)}{\sqrt{m^2-\Box}}G_c(x)=\delta^4(x), \qquad (62)$$

$$\left(\Box=-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial\vec{x}^2}\right)$$

leads to the description of the generalized causal Green function for square-root Klein-Gordon equation, where

$$\begin{aligned}
G_c(x) &= \frac{1}{\sqrt{m^2-\Box}}\delta^4(x) \qquad (63)\\
&= \frac{1}{m\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{\Gamma(n+1/2)}{m^{2n}}(\Box)^n\delta^4(x)
\end{aligned}$$

is the particular case of Efimov's nonlocal or generalized function [9] describing a nonlocal or extended object. This object is distributed in a domain determined by the length

$$L=\frac{\hbar}{mc}.$$

It is obviously that the plane wave $\psi(x) = \frac{1}{(2\pi)^{3/2}} e^{ipx}$ ($px = p_0 x^0 - \overrightarrow{p}\overrightarrow{x}$) satisfies the square-root differential equation

$$\sqrt{m^2 - \Box} e^{ipx} = \frac{m^2 - p^2}{\sqrt{m^2 - p^2}} e^{ipx} = 0, \qquad (64)$$

if $m^2 = p_0^2 - \overrightarrow{p}^2$, where we have used the formula (63) with the change $\delta^4(x) \Rightarrow e^{ipx}$.

Appendix B

It turns out that in our scheme, a new force appears, we call it a fifth force or a dark force due to exchange of square-root or dark matter particles with the propogator $1/\sqrt{m^2 + \overrightarrow{p}^2}$ in the momentum space in the static limit. We know that the Coulomb and Yukawa potentials U_C, U_γ are related with the photon and scalar particles propogators in the static limit by the following formulas:

$$U_C(r) = \frac{e}{(2\pi)^3} \int d^3p e^{i\overrightarrow{p}\overrightarrow{r}} \frac{1}{\overrightarrow{p}^2} = \frac{e}{4\pi r}, \qquad (65)$$

$$U_\gamma(r) = \frac{g}{(2\pi)^3} \int d^3p e^{i\overrightarrow{p}\overrightarrow{r}} \frac{1}{m^2 + \overrightarrow{p}^2} = \frac{g}{4\pi} \frac{e^{-mr}}{r} \qquad (66)$$

Then by analogous with these formulas, we obtain a new potential

$$U_D(r) = \frac{\lambda}{(2\pi)^3} \int d^3p e^{i\overrightarrow{p}\overrightarrow{r}} \frac{1}{\sqrt{m^2 + \overrightarrow{p}^2}} = \frac{\lambda}{2\pi^2} \frac{m}{r} K_1(mr), \qquad (67)$$

where $K_1(z)$ is the Mac Donald function, e, g and λ are some constants. Asymptotic behaviour of this potential takes the form

$$U_D(r) = \begin{cases} \frac{\lambda}{4\pi^2} m^2 \ln \frac{C_z}{2} & z = mr \to 0 \\ \frac{\lambda}{4\pi^2} \frac{m^2}{z} \sqrt{\frac{\pi}{2z}} e^{-z}, & z = mr \to \infty \end{cases} \qquad (68)$$

$C = 0.57721566490\ldots$. It means that a dark particle potential is short distance like the Yukawa one.

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