Wave Solutions of an a-cut Uniaxially Crystal

B. Daariimaa, G. Ochirbat, J. Khatanbaatar, and P. Munkhbaatar*
Theoretical Physics Laboratory, Department of Physics, School of Science, National University of Mongolia, Ulaanbaatar 210646, Mongolia

We consider the problem of a plane wave refraction on an a-cut uniaxially crystal. We show that ordinary and extraordinary waves in Kerr medium could be the plane wave with amplitude and wavevector independent of coordinates. The another result is that the problem of polarization of the isoradial wave in a biaxial crystal is reformulated. The driven equation is slightly different the equation given in Ref.[2]. We solved problem for eigenvalues and eigenvectors of polarization of the electric field for the isoradial wave.

PACS numbers: 81.05.Xj, 42.65.-k, 42.65.Hw

I. INTRODUCTION

Birefringence is used in many optical devices which also plays an important role in second harmonic generation and other nonlinear optical components, as the crystals used for this purpose are almost always birefringent and nonlinear[1]. If the intensity of light propagating through the nonlinear medium is sufficiently high, the refractive index of the medium depends on the intensity of propagating field. The effect is called Kerr effect.

In this paper, we focused on plane electromagnetic (EM) wave propagation in an anisotropic Kerr medium. We consider the intensity profile of refracted wave in uniaxial crystal in II section. In next section problem of polarization of the isoradial wave in a biaxial crystal is reformulated as Ref.[2]. We solved eigenvalues and eigenvectors problem for polarization of electric field of the isoradial wave.

II. DOUBLE REFRACTION AT A BOUNDARY OF AN a-CUT UNIAXIALLY CRYSTAL SURFACE

An important physical consequence for the wave propagation in anisotropic media is double refraction[3]. Poynting vector or energy flux of an electromagnetic field is

$$\mathbf{S} = \frac{1}{2}\,\mathrm{Re}[\mathbf{E}\times\mathbf{H}^*]. \tag{1}$$

The refracted wave, in general, is a mixture of ordinary wave and extraordinary wave, so its electric and magnetic fields are given, respectively, by

$$\mathbf{E} = (C_o\,\mathbf{o}\,e^{-i\mathbf{k}_o\cdot\mathbf{r}} + C_e\,\mathbf{e}\,e^{-i\mathbf{k}_e\cdot\mathbf{r}})e^{i\omega t}, \tag{2}$$

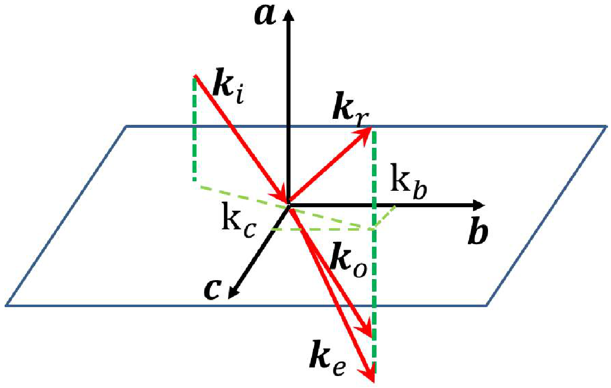

Figure 1: Double refraction at boundary of an a-cut uniaxially crystal surface

$$\mathbf{H} = \frac{1}{\omega\,\mu}\left\{C_o e^{-i\mathbf{k}_o\cdot\mathbf{r}}\,\mathbf{k}_o\times\mathbf{o} + C_e e^{-i\mathbf{k}_e\cdot\mathbf{r}}\,\mathbf{k}_e\times\mathbf{e}\right\}e^{i\omega t}. \tag{3}$$

Here $\mathbf{k}_o$ ($\mathbf{k}_e$) and $\mathbf{o}$ ($\mathbf{e}$) are wavevector and unit vector of polarization of ordinary (extraordinary) wave, respectively.

The orientation of the crystal, the incident wave and refracted wave is shown in Fig. 1. For the case of an a-cut uniaxially crystal surface[3], the intensity of the wave is

$$I = \mathbf{a}\cdot\mathbf{S}. \tag{4}$$

Using (2) and (3), we can find three terms of the intensity:

$$I_o = \frac{C_o^2}{2\omega\mu}\,\mathbf{a}\cdot\mathbf{o}\times(\mathbf{k}_o\times\mathbf{o}), \tag{5}$$

$$I_e = \frac{C_e^2}{2\omega\mu}\,\mathbf{a}\cdot\mathbf{e}\times(\mathbf{k}_e\times\mathbf{e}), \tag{6}$$

*Electronic address: `p_munhkbaatar@yahoo.com`

$$\begin{aligned} I_{oe} &= \frac{C_o C_e}{2\omega\mu}\mathrm{Re}\{\mathbf{a}\cdot\mathbf{o}\times(\mathbf{k}_e\times\mathbf{e})e^{i(\mathbf{k}_e-\mathbf{k}_o)\cdot\mathbf{r}} \\ &+ \mathbf{a}\cdot\mathbf{e}\times(\mathbf{k}_o\times\mathbf{o})e^{-i(\mathbf{k}_e-\mathbf{k}_o)\cdot\mathbf{r}}\} \\ &= \frac{C_o C_e}{2\omega\mu}\{\mathbf{a}\cdot\mathbf{o}\times(\mathbf{k}_e\times\mathbf{e}) \\ &+ \mathbf{a}\cdot\mathbf{e}\times(\mathbf{k}_o\times\mathbf{o})\}\cos[(\mathbf{k}_e-\mathbf{k}_o)\cdot\mathbf{r}]. \quad (7) \end{aligned}$$

The last (7) term is the interference between ordinary and extraordinary waves. This equals to zero: $I_{oe} = \mathbf{a}\cdot\mathbf{S}_{oe} = 0$ or

$$\mathbf{a}\cdot\mathbf{o}\times(\mathbf{k}_e\times\mathbf{e}) + \mathbf{a}\cdot\mathbf{e}\times(\mathbf{k}_o\times\mathbf{o}) = 0. \quad (8)$$

Thus, the intensity of transmitted wave is the sum of intensities of ordinary and extraordinary waves which is independent on coordinates and time.

If the refractive index of the principal axis is dependent on an intensity (Kerr medium), i.e both ordinary and extraordinary indexes independent on space coordinates and time, there could be the self-consistent plane wave with amplitude and wavevector independent on coordinate.

When directions of the phase velocities of ordinary and extraordinary waves are matched (isonormal wave, $\mathbf{k}_o//\mathbf{k}_e$), the interference term in eq.(7) of energy flux is non zero. For instance,

$$S_{oe} = -\frac{C_o C_e}{2\omega\mu}\,\mathbf{o}\,(\mathbf{k}_o\cdot\mathbf{e})\,\cos((\mathbf{k}_e-\mathbf{k}_o)\cdot\mathbf{r}). \quad (9)$$

If refractive indexes on the principal axis are depending on an intensity, the refractive indexes depend on coordinated because the interference term dependent on coordinates. Thus, there cannot be an isonormal plane wave with constant amplitude and wave vector.

III. PLANE WAVE IN BIAXIAL CRYSTAL

Let us consider **E**, **D** and **H** real vectors. A wave packet can be viewed as a linear superposition of many monochromatic plane waves, each with a definite frequency ω and wavevector **k**. Each plane wave component satisfies the following Maxwell's equations in momentum space. Then Maxwell's equations are:

$$\mathbf{k}\times\mathbf{H} = -\omega\mathbf{D}, \quad \mathbf{k}\times\mathbf{E} = \omega\mu_0\mathbf{H}. \quad (10)$$

The energy flux in the plane wave, i.e. its Poynting vector is

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = \frac{1}{\omega\mu_0}\mathbf{E}\times(\mathbf{k}\times\mathbf{E}) = \frac{1}{\omega\mu_0}\left[\mathbf{k}E^2 - \mathbf{E}\,(\mathbf{k}\cdot\mathbf{E})\right]. \quad (11)$$

The energy flux in biaxial crystal isn't in same direction of the wavevector. On the other hand the energy density is

$$w = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}) = \frac{1}{\omega}\mathbf{k}\cdot\mathbf{S}. \quad (12)$$

The refractive vector of the ray is given by

$$\mathbf{p} = \frac{\mathbf{S}}{c\,w}. \quad (13)$$

As well know connection,

$$\mathbf{k} = \frac{\omega}{c}\mathbf{n}. \quad (14)$$

Here **n** is the refractive vector of the phase. Using these definitions of **p** and **n**, we can get as following form of the eq.(12):

$$\mathbf{p}\cdot\mathbf{n} = 1. \quad (15)$$

p is in same direction of **s**:

$$\mathbf{p}\cdot\mathbf{E} = 0, \qquad \mathbf{p}\cdot\mathbf{H} = 0. \quad (16)$$

If (10) equations are multiplied by the cross product of **p** vector, using equations of (14)-(16), we have

$$\mathbf{H} = c\,\mathbf{p}\times\mathbf{D}, \qquad \mathbf{E} = -c\,\mu_0\mathbf{p}\times\mathbf{H}. \quad (17)$$

If we use these notations $\mathbf{H}' \equiv \mu_0 c\mathbf{H}$, $\mathbf{D}' \equiv \varepsilon\mathbf{E}$, $\varepsilon \equiv \mu_0 c^2 n^2$, (17) equations become

$$\mathbf{E} = -\mathbf{p}\times\mathbf{H}', \qquad \mathbf{H}' = \mathbf{p}\times\mathbf{D}'. \quad (18)$$

Now, lets assume that $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are unit vectors of the principal axis of dielectric tensor, the tensor is given by the outer product of these vectors ($\varepsilon_1 < \varepsilon_2 < \varepsilon_3$):

$$\varepsilon = \varepsilon_1\mathbf{u}_1\mathbf{u}_1 + \varepsilon_2\mathbf{u}_2\mathbf{u}_2 + \varepsilon_3\mathbf{u}_3\mathbf{u}_3. \quad (19)$$

Furthermore, we will establish the following two vectors using $\mathbf{u}_1, \mathbf{u}_3$:

$$\mathbf{c}_1 = k_1\mathbf{u}_1 + k_3\mathbf{u}_3, \qquad \mathbf{c}_2 = -k_1\mathbf{u}_1 + k_3\mathbf{u}_3. \quad (20)$$

Here,

$$k_1 = \sqrt{\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_3-\varepsilon_1}}, \qquad k_3 = \sqrt{\frac{\varepsilon_3-\varepsilon_2}{\varepsilon_3-\varepsilon_1}}.$$

So, let us express the eq.(19) by $\mathbf{c}_1$, $\mathbf{c}_2$ vectors:

$$\varepsilon = \varepsilon_2\,\mathbb{I} + \frac{\varepsilon_3-\varepsilon_1}{2}\left(\mathbf{c}_1\mathbf{c}_2 + \mathbf{c}_2\mathbf{c}_1\right). \quad (21)$$

Here $\mathbb{I}$ is unit operator.

Polarization isoradial wave: According to the literature [2], an equation for electric field of isoradial wave have been written as following form:

$$\begin{aligned} \{1 - p^2\varepsilon_2 &- \frac{\varepsilon_3-\varepsilon_1}{2}\left([\mathbf{p}^\times\mathbf{c}_1][\mathbf{p}^\times\mathbf{c}_2]\right. \\ &+ \left.[\mathbf{p}^\times\mathbf{c}_2][\mathbf{p}^\times\mathbf{c}_1]\right)\}\,\mathbf{E} = 0. \quad (22) \end{aligned}$$

Here $\mathbf{c}_1, \mathbf{c}_2$ vectors are defined by (20). And $\mathbf{p}^\times$ is the antisymmeter tensor of the vector **p**. Using this tensor, the cross product of two vectors can be written as $\mathbf{p}^{\times}\mathbf{c} \equiv \mathbf{p}\times\mathbf{c}$.

$$\mathbf{p}^{\times}\mathbf{p}^{\times} = \mathbf{p}\,\mathbf{p} - p^2,\ \mathbf{c}\mathbf{p}^{\times} = -\mathbf{p}^{\times}\mathbf{c} = -\mathbf{p}\times\mathbf{c}. \tag{23}$$

We couldn't understand completely the eq.(22). Thus we decided to solve the problem from the begin. From the eq.(18)

$$\mathbf{E} = -\mathbf{p}^{\times}\mathbf{H}' = -\mathbf{p}^{\times}\mathbf{p}^{\times}\mathbf{D}', \qquad \mathbf{D}' = \varepsilon\mathbf{E} \tag{24}$$

or

$$\mathbf{E}\ +\ \mathbf{p}^{\times}\mathbf{p}^{\times}\varepsilon\mathbf{E} = 0. \tag{25}$$

If we use the eq.(21), we have

$$\mathbf{p}^{\times}\mathbf{p}^{\times}\varepsilon\mathbf{E} = \varepsilon_2\mathbf{p}^{\times}\mathbf{p}^{\times}\mathbf{E} + \frac{\varepsilon_3-\varepsilon_1}{2}(\mathbf{p}^{\times}\mathbf{p}^{\times}\mathbf{c}_1(\mathbf{c}_2\cdot\mathbf{E}) + \mathbf{p}^{\times}\mathbf{p}^{\times}\mathbf{c}_2(\mathbf{c}_1\cdot\mathbf{E})). \tag{26}$$

We can change two terms within the parentheses:

$$\begin{aligned}\mathbf{p}^{\times}\mathbf{p}^{\times}\mathbf{c}_1(\mathbf{c}_2\cdot\mathbf{E}) = (\mathbf{E}\times(\mathbf{p}\times\mathbf{c}_2))\times(\mathbf{p}\times\mathbf{c}_1) &= (\mathbf{p}\times\mathbf{c}_2)\,((\mathbf{p}\times\mathbf{c}_1)\cdot\mathbf{E}) - ((\mathbf{p}\times\mathbf{c}_1)\cdot(\mathbf{p}\times\mathbf{c}_2))\,\mathbf{E},\\ \mathbf{p}^{\times}\mathbf{p}^{\times}\mathbf{c}_2(\mathbf{c}_1\cdot\mathbf{E}) = (\mathbf{E}\times(\mathbf{p}\times\mathbf{c}_1))\times(\mathbf{p}\times\mathbf{c}_2) &= (\mathbf{p}\times\mathbf{c}_1)\,((\mathbf{p}\times\mathbf{c}_2)\cdot\mathbf{E}) - ((\mathbf{p}\times\mathbf{c}_2)\cdot(\mathbf{p}\times\mathbf{c}_1))\,\mathbf{E}.\end{aligned} \tag{27}$$

In here, we used $\mathbf{p}(\mathbf{c}_i\cdot\mathbf{E}) = \mathbf{E}\times(\mathbf{p}\times\mathbf{c}_i)$ because of $\mathbf{p}\cdot\mathbf{E} = 0$. Thus, the eq.(25) is now:

$$\left\{1 - p^2\varepsilon_2 - 2\frac{\varepsilon_3-\varepsilon_1}{2}[\mathbf{p}^{\times}\mathbf{c}_1]\cdot[\mathbf{p}^{\times}\mathbf{c}_2] + \frac{\varepsilon_3-\varepsilon_1}{2}([\mathbf{p}^{\times}\mathbf{c}_1][\mathbf{p}^{\times}\mathbf{c}_2] + [\mathbf{p}^{\times}\mathbf{c}_2][\mathbf{p}^{\times}\mathbf{c}_1])\right\}\mathbf{E} = 0. \tag{28}$$

The eq.(28) is quite different from the eq.(22). First, the third term is new. Second, fourth term has opposite sign. But, the problem of eigenvectors and eigenvalues of this equation procedure is same as finding eigenvectors and eigenvalues of isonormal waves for magnetic field. Further, we have

$$Q = [\mathbf{p}^{\times}\mathbf{c}_1][\mathbf{p}^{\times}\mathbf{c}_2] + [\mathbf{p}^{\times}\mathbf{c}_2][\mathbf{p}^{\times}\mathbf{c}_1].$$

Also, eigenvalues of this operator are

$$\lambda_{\pm} = [\mathbf{p}^{\times}\mathbf{c}_1]\cdot[\mathbf{p}^{\times}\mathbf{c}_2] \pm \sqrt{[\mathbf{p}^{\times}\mathbf{c}_1]^2[\mathbf{p}^{\times}\mathbf{c}_2]^2}. \tag{29}$$

Eigenvectors are

$$\sqrt{[\mathbf{p}^{\times}\mathbf{c}_2]^2}\mathbf{p}^{\times}\mathbf{c}_1 \pm \sqrt{[\mathbf{p}^{\times}\mathbf{c}_1]^2}\mathbf{p}^{\times}\mathbf{c}_2. \tag{30}$$

For the eigenvalue of the refractive vector of the ray, the eq.(28) is now

$$1 - p^2\varepsilon_2 - 2\frac{\varepsilon_3-\varepsilon_1}{2}[\mathbf{p}^{\times}\mathbf{c}_1]\cdot[\mathbf{p}^{\times}\mathbf{c}_2] + \frac{\varepsilon_3-\varepsilon_1}{2}\lambda_{\pm} = 0. \tag{31}$$

If we write the refractive vector of the ray $\mathbf{p} = \frac{\mathbf{s}}{s}$ using $\mathbf{s}$ unit vector, substitution $\mathbf{p}$ into the eq.(31) give us eigenvalues for s:

$$s_{\pm}^2 = \varepsilon_2 + \frac{\varepsilon_3-\varepsilon_1}{2}([\mathbf{s}^{\times}\mathbf{c}_1]\cdot[\mathbf{s}^{\times}\mathbf{c}_2] \mp \sqrt{[\mathbf{s}^{\times}\mathbf{c}_1]^2[\mathbf{s}^{\times}\mathbf{c}_2]^2}). \tag{32}$$

Corresponding eigenvectors are

$$\mathbf{E}_{\pm} = A_{\pm}\mathbf{e}_{\pm}. \tag{33}$$

Here A_+, A_- is arbitrary constant,

$$\mathbf{e}_{\pm} = \sqrt{[\mathbf{s}^{\times}\mathbf{c}_2]^2}\mathbf{s}^{\times}\mathbf{c}_1 \pm \sqrt{[\mathbf{s}^{\times}\mathbf{c}_1]^2}\mathbf{s}^{\times}\mathbf{c}_2. \tag{34}$$

We have shown that the driven equation for $\mathbf{E}$ electric fields of isoradial waves is slightly different the equation given in Ref.[2]. We solved the problem for eigenvalues and eigenvectors of the driven equation.

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