Alternative modification of KBM for solving equations of TE wave spreading in depth of kerr substrate with linear absorbtion

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KBM method, specialy modified for solving a pair of first order differential equations for TM wave spreading in depth of kerr substrate with linear absorbtion, is used to solve the same problem in the case of TE polarization. Solution, obtained by this alternative modification of KBM, is in full agreement with the solution to Helmgolts's equation, previously found by another modification of KBM. It is shown that the one of asymptotic series obtained with the method is convergent.

Papers [1-5] are devoted to solving problems of propagation of stationary light waves in media with absorption. In [5] methods of Krilov-Bogolubov-Metropolsky is modified to solve a set of two Maxwell's equations in propagation problem of stationary optical TM waves in nonlinear substrate with linear absorption. Our purpose is to apply the scheme used in [5] to the same problem of TE waves.

We consider a nonlinear differential equation of the form

$$\frac{d^2 e(z)}{dz^2} + (-\beta^2 + \in)e(z) = 0, \qquad (1)$$

$$\in = \varepsilon + i\delta + \alpha |e(z)|^2,$$

Where $\delta = \frac{4\pi\sigma}{\omega}$, ε is a dielectric constant, α is a constant, σ is a conductivity.

For the linear case , we get solution of (1) in the form

$$e(z) = e_0 = u_0 \exp(i\eta z + i\psi_0).$$

Where u_0 and ψ_0 are constants, $\eta = q + i\lambda, q^2 - \lambda^2 = \gamma, \delta = 2\lambda q$ and $\lambda > 0$.

An idea of the method[5]

Let us find a solution of (1) in the form

$$e = u \cdot \exp(i\psi)$$

(3)

for the case of $\alpha \neq 0$.

Then u and ψ can be determined by the following equations

$$\frac{du}{dz} = -\lambda u + \alpha u_1(u) + \alpha^2 u_2(u) + \cdots,$$

(4)

$$\frac{d\psi}{dz} = q + \alpha v_1(u) + \alpha^2 v_2(u) + \cdots,$$

(5)

Where u_l and v_l (l=1,2,...) are real valued functions of u.

Substituting (3), (4) and (5) into (1') and comparing coefficients of identical powers by α, we find:

$$u_1 = \alpha_1 a^3, \quad \alpha_1 = \frac{2\lambda}{2q^2 + 8\lambda^2},$$

(6.1)

$$v_1 = \beta_1 a^2, \quad \beta_1 = \frac{q}{2q^2 + 8\lambda^2},$$

(6.2)

$$u_2 = \alpha_2 a^5, \quad \alpha_2 = \frac{4\alpha_1 \beta_1 q - \lambda(9\alpha_1^2 - 3\beta_1^2)}{18\lambda^2 + 2q^2},$$

(6.3)

$$v_2 = \beta_2 a^4, \quad \beta_2 = \frac{-(6\lambda^2 \alpha_2 - \lambda(3\alpha_1^2 - \beta_1^2))}{2\lambda q},$$

(6.4)

There are relationship between α_n and β_n by

$$\sum_{j=1}^{n-1}(2j+1)\alpha_j \alpha_{n-j} - (2n+2)\lambda\alpha_n - \overset{(2)}{2q}\beta_n - \sum_{j=1}^{n-1}\beta_j \beta_{n-j} = 0,$$

(7.1)

$$2\sum_{j=1}^{n-1}(n+1-j)\alpha_j \beta_{n-j} + 2q\alpha_n - (2n+2)\lambda\beta_n = 0$$

(7.2)

Where α_1 and β_1 are determined by formula(6.1) and (6.2)

It has been shown that the series (expansion) (4) and (5) would be convergent for the case a) $\lambda = 0$ and b) $q = 0$, the limits of which are coinsided to analitytic solution.

Now we will solve problem(1) in the same way as in [5].

Equation(1) can be represented in its equivalent form

$$\frac{dH}{dz} = (\in - \beta^2)e(z)$$

$$\frac{de}{dz}=-H.$$

Then for $\alpha=0$ formula (8) and (9) imply

$$\frac{dH}{dz}=(\varepsilon+i\delta-\beta^2)e,$$

(10)

$$\frac{de}{dz}=-H$$

(11)

The solution to (10) and (11) can be found in the form

$$\begin{cases} H=h_0e^{i\eta z} \\ e=u_0e^{i\eta z} \end{cases}$$

(12)

Substituting (12) into equality (10) and (11) we obtain the following homogeneous system of equations

$$\begin{cases} i\eta h_0=(\varepsilon+i\delta-\beta^2)u_0 \\ i\eta u_0=-h_0 \end{cases}$$

(13)

The condition that the system of equations has a nontrivial solution is:

$$\det\begin{vmatrix} i\eta & -(\varepsilon+i\delta-\beta^2) \\ 1 & i\eta \end{vmatrix}=0$$

That is $\eta^2=\varepsilon-\beta^2+i\delta$. .If we assume that, $\eta=q+i\lambda$, , then we have

$q^2-\lambda^2=\varepsilon-\beta^2=\gamma,\quad \delta=2q\lambda.$

Whence

$$\left(\frac{\delta}{2\lambda}\right)^2-\lambda^2=\gamma \text{ or } \lambda^2=-\frac{1}{2}\gamma+\frac{\sqrt{\gamma^2+\delta^2}}{2}.$$

So, we have a general solution of the form

$$H=h_0^+\exp(-\lambda z+iqz+i\psi_0)+h_0^-\exp(\lambda z-iqz+i\psi_0),$$

$$e=u_0^+\exp(-\lambda z+iqz+i\psi_0)+u_0^-\exp(\lambda z-iqz+i\psi_0).$$

We set $h_0^-=0$ and $u_0^-=0$, , because the propagating along to depth of substrate and damping wave is considered.

Let us find a solution to (8) and (9) in the form

$$\begin{cases} H=h\cdot e^{i\psi z} \\ e=u\cdot e^{i\psi z} \end{cases}$$

(14)

with $\alpha\neq 0$, ,where u is a real number, h is a complex one.

Then u,ψ, and h can be determined by the following equations

$$\frac{du}{dz}=-\lambda u+\alpha u_1+\alpha^2u_2+...,(9)$$

(15)

$$\frac{d\psi}{dz}=q+\alpha v_1+\alpha^2v_2+...,$$

(16)

$$h=h_0+\alpha h_1+\alpha^2h_2+...,$$

(17)

Where u_i and v_i $(i=1,2,...)$ are real valued function of u, and h_i $(i=1,2,...)$ are complex valued functions of u.

Taking into consideration equality (15)-(17), we obtain from (14)

$$\frac{dH}{dz}=\{(\dot h_0+\alpha\dot h_1+...)(-\lambda u++\alpha u_1+...)+$$
$$+i(h_0+\alpha h_1+...)(q++\alpha v_1+...)\}e^{i\psi},$$

(18)

$$\frac{de}{dz}=\{(-\lambda u++\alpha u_1+...)+iu(q+\alpha v_1+...)\}e^{i\psi},$$

(19)

Where $\dot h_i$ $(i=1,2,...)$ are derivatives of h_i with respect to u.

Substituting (18) and (19) into (8) and (9) and comparing coefficients of identical powers by α , the functions u_i,v_i and h_i $(i=1,2,...)$ are found in the following ways.

The "zero" power of the α implies

$$\begin{cases} -\lambda u\dot h_0+iqh_0=(\varepsilon+i\delta-\beta^2)u \\ -\lambda u+iqu=-h_0, \end{cases}$$

that is

$$h_0=(\lambda-iq)u.$$

The first power of the is:

$$\begin{cases} -\lambda u\dot h_1+\dot h_0u_1+ih_0v_1+iqh_1=u^3 \\ u_1+iuv_1=-h_1. \end{cases}$$

(21)

From equality (20) , we get

$$\dot h_0u_1+ih_0v_1=(\lambda-iq)(u_1+iuv_1).$$

(22)

And with (21), bearing in mind (22), we obtain

$$-\lambda u\dot h_1+(2qi-\lambda)h_1=u^3.$$

(23)

Let us find a particular solution to (23) in the form

$$h_1=-v_1u^3.$$

Then (23) implies

$$v_1 = \frac{1}{4\lambda - 2qi}.$$

(24)

Furthermore, assuming that $v_1 = \alpha_1 + i\beta_1$,we obtain

$$\alpha_1 = \frac{2\lambda}{2q^2 + 8\lambda^2},$$

(25)

$$\beta_1 = \frac{q}{2q^2 + 8\lambda^2}.$$

(26)

Then, the second equation in the system of equations (21) implies

$$u_1 = \alpha_1 u^3.$$

(27)

$$v_1 = \beta_1 u^2.$$

(28)

Finally, we have

$$\sum_{j=0}^{n-1} \dot{h}_j u_{n-j} - \lambda u \dot{h}_n + i\left(\sum_{j=0}^{n-1} h_j v_{n-j} + h_n q\right) = 0$$

(29)

$$u_n + iuv_n = -h_n \qquad (30)$$

for the n-th power of the α.

From equality (20), we also get

$$\dot{h}_0 u_n + ih_0 v_n = (\lambda - iq)(u_n + iuv_n). \qquad (31)$$

With (29)-(30), bearing in mind (31), we obtain:

$$\sum_{j=1}^{n-1} \dot{h}_j u_{n-j} + (2qi - \lambda)h_n - \lambda u \dot{h}_n + i\sum_{j=1}^{n-1} h_j v_{n-j} = 0.$$

(32)

Finding a particular solution to (32) in the form

$$h_n = -v_n u^{2n+1},$$

(33)

For v_n we get the following recurrent relation

$$\sum_{j=1}^{n-1}(2j+1)v_j\alpha_{n-j} + (2qi - (2n+2)\lambda)v_n + \qquad (34)$$

$$i\sum_{j=1}^{n-1} v_j \beta_{n-j} = 0$$

It is easy to show that the formula (34) coincides to (7.1), (7.2) , if we assume $v_j = \alpha_j + \beta_j \ (j = 1, 2, \dots n)$.

Thus we have shown that the solutions found by different methods, are the same.

Convergence

At this stage we able to find any coefficients of the series (17), using (33) and recurrent formula (34)

$$h = -u\left\{(iq - \lambda) + v_1\left[u^2\alpha\right] + \dots + v_n\left[u^2\alpha\right]^n + \dots\right\},$$

(35)

Where v_j are given by (33).

If the series (17) is convergent in same region then the function e(z) will be defined by the integral

$$e = -\int h \cdot e^{iqz} dz.$$

(36)

In other words we have found not only asymptotic but also analytical solution.

For this purpose, let us prove the following lemma.

Lemma

$$\sum_{j=1}^{n-1} k_j k_{n-j} = -2k_n$$

(37)

Where

$$k_n = \frac{1}{2}\left(\frac{1}{2} - 1\right)\dots\left(\frac{1}{2} - (n-1)\right)(n!)^{-1}, \quad (n \geq 1) \quad (38)$$

Proof

Using Taylor's formula for the function $(1+\alpha)^{1/2}$ *, in a neighborhood of the point* $\alpha = 0$ *and bearing in mind (38)., we obtain:*

$$(1+\alpha)^{\frac{1}{2}} - 1 = k_1\alpha + k_2\alpha^2 + \dots + k_n\alpha^n + \dots$$

(39)

Squaring (39), we get

$$\left\{(1+\alpha)^{\frac{1}{2}} - 1\right\}^2 = k_1^2\alpha^2 + (k_1k_2 + k_2k_1)\alpha^3 + \dots$$

$$+ \left(\sum_{j=1}^{n-1} k_j k_{n-j}\right)\alpha^n + \dots$$

(40)

It is easy to show that

$$2\left(1 + \frac{1}{2}\alpha - (1+\alpha)^{\frac{1}{2}}\right) =$$

$$2\left\{1 + k_1\alpha - \left[1 + k_1\alpha + \dots + k_n\alpha^n + \dots\right]\right\}.$$

(41)

Using the equation $\left\{(1+\alpha)^{\frac{1}{2}} - 1\right\}^2 = 2\left(1 + \frac{1}{2}\alpha - (1+\alpha)^{\frac{1}{2}}\right)$ *and comparing formulas (40) and (41) , we have*

$$\sum_{j=1}^{n-1} k_j k_{n-j} = -2k_n.$$

Thus the proof of the lemma is complete.

Now we consider the following sequence.

$$c_1 = 1, \quad c_n = \sum_{j=1}^{n-1} c_j c_{n-j}. \tag{42}$$

Corollary

$$c_n = (-1)^{n-1} 2^{2n-1} k_n = (-1)^n 2^{2n-1} \frac{\frac{1}{2}(\frac{1}{2}-1)...(\frac{1}{2}-(n-1))}{n!}. \tag{43}$$

Proof

We shall prove by induction

$$c_1 = (-1)^0 2^1 \cdot \frac{1}{2} = 1,$$

$$c_n = \sum_{j=1}^{n-1} c_j c_{n-j} = \sum_{j=1}^{n-1} (-1)^{j-1} 2^{2j-1} k_j (-1)^{n-j} 2^{2(n-j)-1} k_{n-j} =$$

$$= (-1)^{n-2} 2^{2n-2} \sum_{j=1}^{n-1} k_j k_{n-j} = (-1)^n 2^{2n-1} k_n.$$

Consider sequence

$$\mu_1 = \frac{1}{2\lambda}, \quad \mu_n = \frac{1}{2\lambda} \sum_{j=1}^{n-1} \mu_j \mu_{n-j}. \tag{44}$$

It is easily proved by induction that

$$\mu_n = \frac{c_n}{2\lambda^{2n-1}} = (-1)^n \frac{\frac{1}{2}(\frac{1}{2}-1)...(\frac{1}{2}-(n-1))}{\lambda^{2n-1} n!}. \tag{45}$$

Using the substitution

$$\alpha_j = \frac{v_j + v_j^*}{2},$$

$$\beta_j = \frac{v_j - v_j^*}{2}$$

We rewrite (34) in the complex form

$$v_n = \frac{\sum_{j=1}^{n-1} (j(v_{n-j} + v_{n-j}^*) + v_{n-j}) v_j}{-2qi + (2n+2)\lambda}.$$

Estimating by module we have

$$|v_n| \le \frac{\sum_{j=1}^{n-1} (2j+1)|v_j||v_{n-j}|}{(2n+2)\lambda} = \frac{1}{2\lambda} \sum_{j=1}^{n-1} |v_j||v_{n-j}|,$$

$$|v_1| = \frac{1}{\sqrt{4q^2 + 16\lambda^2}} \le \frac{1}{4\lambda} < \frac{1}{2\lambda} = \mu_1.$$

It is easily shown that

$$|v_n| < \mu_n \text{ for any } n = 1, 2...$$

Thus we have found major series with coefficients, which have estimated the series (35) by module.

The sufficient condition that the series (35) to be convergent is

$$\frac{\alpha u^2}{\lambda} < 1. \tag{46}$$

In fact, according to Dalamber's criterion, we have

$$\lim_{n\to\infty} \frac{\mu_{n+1} \left[u^2\alpha\right]^{n+1}}{\mu_n \left[u^2\alpha\right]^n} = \frac{u^2\alpha}{\lambda^2} \lim_{n\to\infty} \frac{n - \frac{1}{2}}{(n+1)} = \frac{u^2\alpha}{\lambda^2}.$$

. Conclusion

KBM method, specialy modified for solving a pair of first order differential equations for TM wave spreading in depth of kerr substrate with linear absorbtion, is used to solve the same problem in the case of TE polarization. Solution, obtained by this alternative modification of KBM, is in full agreement with the solution to Helmgolts's equation, previously found by another modification of KBM. It is shown that the one of asymptotic series obtained with the method is convergent.

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Дүгнэлт

Шугаман шингээлттэй керр сууръ орчны гүн рүү тарах ТМ долгионы нэгдүгээр эрэмбэ бүхий хос дифференциаль тэгшитгэл бодоход тохируулсан КБМ аргыг ТЕ туйлшралтай тохиолд мөнхүү бодлогыг бодоход хэрэглэв. КБМ-ийн энэхүү хувилбараар олсон шийд нь Гельмгольцийн тэгшитгэлийн КБМ-ийн өөр хувилбараар урьд олсон шийдтэй яг тохирч байлаа. Энэ хувилбараар олсон нэг асимптотик цуваа нийлнэ гэж харуулав