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Exact Solution of Two-Axis-Twisting Hamiltonian in Bose-Einstein Condensate

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In this Paper, we have dealt with two-axis twisting Hamiltonian for spin squeezing, which is experimentally implementable in Bose-Einstein Condensate, and have found an exact solution by two alternative methods. Many other important nonlinear Hamiltonians could also be solved exactly in a similar way.

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I. INTRODUCTION

It is well known that nonlinear Hamiltonians play important role in preparing and manipulating quantum correlations or entanglement. For instance, nonlinear Hamiltonians of collective spin systems, namely one-axis and two-axis twisting models, are suggested for demonstrating a spin squeezing concept [1]. Astonishing experimental advances in Bose-Einstein-Condensation (BEC) [2–5] and ion traps occurred in the last few years open interest to many nonlinear collective Hamiltonians including these two nonlinear model of spin squeezing. One-axis twisting model has an analytical solution but, at our knowledge, no exact solution to two-axis twisting model is reported so far. Numerical calculation is also fairly difficult due to its nested complexity of commutation relation between two nonlinear terms of the Hamiltonian.

In this Paper, an exact solution of two-axis twisting model for spin squeezing will be given. This Paper is organized as follows. In introduction part, we briefly discuss about BEC, especially Hamiltonians of two-mode BEC, in order to make clear the basic physics of many interesting nonlinear Hamiltonians like one-axis twisting and also two-axis model of spin squeezing that can be implemented in such a system. Since our simple purpose is to demonstrate an exact solution for two-axis twisting Hamiltonian which covers the main part of the Paper, we omit important discussions regarding on physical processes and applications of BEC, but there are many good review articles [6–8]. We will be interested only in the evolution of entanglement and squeezing of an initial CSS under two-axis twisting Hamiltonian based on the results of Ref. [9]; numerical results are given the final part. Our approach to the problem may also be applicable for other interesting but not yet solved nonlinear Hamiltonians implementable in BEC.

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II. SHORT INTRODUCTION TO BOSE-EINSTEIN CONDENSATION

BEC is predicted 1924 by A.Einstein on the basis of ideas of S.Bose concerning photons: In a system of particles obeying Bose statistics and whose total number is conserved, there should be a temperature below which a finite fraction of all the particles "condensate" into the same one-particle state. In such a condensed state all atoms are absolutely identical and all together behaves like a "super molecule". Einstein's original prediction was for noninteracting gas, but after the observation by Kapitsa the superfluidity in liquid ^{4}He below the critical temperature of 2.17 in 1925, F. London suggested that despite the strong interatomic interactions BEC was indeed occurring in this system and was responsible for superfluidity.

Finally, BEC was created in 1995 on vapors of alkalis (rubidium, sodium and lithium etc) trapped by magnetic fields [2]. These experiments where so remarkable that a small sample of atoms was cooled down to only a few billionth (10^{-9}) of a degree above Absolute Zero by using laser cooling and then by evaporative cooling. Since then BEC becomes so hot topic of experimental and theoretical study that there are several hundred scientific papers already published. And the creators Wolfgang Ketterle of the Massachusetts Institute of Technology (MIT) and Carl Wieman and Eric Cornell of JILA, an interdisciplinary research center in Boulder, Colorado, have won Nobel Prize in Physics, in 2001, for their work in making and understanding BECs.

System. So far the BEC has been realized in ^{87}Rb, ^{23}Na, ^{7}Li which has nuclear spin-$\frac{1}{2}$, and much work have been devoted in implementation on vapors of cesium, potassium, and metastable helium. Recently BEC has been implemented on ^{85}Rb which has nuclear spin-$\frac{3}{2}$. In dilute and cold gases three-body collision are very rare event ($\sim 10^{-29} - 10^{-30}\mathrm{cm}^6\mathrm{sec}^{-1}$), and hence atoms can stay in metastable gas phase before freezing to solid phase for a reasonable time of observing BEC (from a few second to a few minutes), even at such a low temperature (typically a few tens of $nK - \sim 50\mu K$). The reason is that the formation of the solid requires as the first step the recombination of two atoms to form a diatomic molecule, and while this process is certainly exothermic (with formation energies typically $\sim 0.4 - 1.2$eV) it is very slow in the absence of a third atom to carry of the surplus energy, angular momentum etc.

Typical number of cooled atoms are $\sim 10^2 - 10^{10}$and their densities range from $\sim 10^{11}$ to $\sim 5 \times 10^{15}$. Therefore they forms almost macroscopic system directly measurable with optical method. Typically the average distance between atoms is more than then times the range of interatomic force so the gas is very dilute. Therefore two-body interaction plays main role in nonlinear interactions among the particles. Another well-known fact that these gases are highly inhomogeneous has several important consequences. The condensate can possess both momentum space and also coordinate space and allows one to study many important quality like the temperature dependence of the condensate, energy and density distributions, interference phenomena, frequencies of collective excitation and so on.

Definition. Penrose and Onsager [10] have generalized the concept of BEC to interacting Bose systems in 1956. A system of N boson is considered Bose condensed (single condensate) if its single particle density matrix has a single macroscopic (of order N) eigenvalue. The corresponding eigenfunction is identified as the quantum state macroscopically occupied. Hamiltonian of the single BEC is modeled as follows: nonlinear interaction between the particles are basically two particle interaction since three particle interaction is very rare as said above.

It is well known result that low energy scattering of two particles interacting via a central potential $V(r)$ is defined by only one simple parameter called s-wave scattering length a_s which may have either sign depending on the details of the potential (negative for attraction and positive for repelling interactions). In general it depends on chemical and isotopic species, hyperfine indices of the two atom and even on magnetic or laser field and are experimentally determined for every species. The fundamental result for alkali gas system is that the true interaction potential of two indistinguishable atoms of reduced mass m_r may be replaced by a delta function of strength

$$U(\mathbf{x}) = \frac{8\pi\hbar^2 a_s}{m}\delta(\mathbf{x}) \tag{1}$$

and sometimes called as δ-pseudopotential or binary, contact interaction approximation.

Under this approximation, second quantized Hamiltonian for BEC can be written in the form

$$H = \int d\mathbf{x}\hat{\psi}^\dagger(\mathbf{x})\left[-\frac{\hbar^2}{2m}\nabla^2 + V_T(\mathbf{x})\right]\hat{\psi}(\mathbf{x}) + \frac{1}{2}g\int d\mathbf{x}\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{x}) \tag{2}$$

where $V_T(\mathbf{x})$ is a trap potential, $\hat{\psi}^\dagger(\mathbf{x})$ and $\hat{\psi}(\mathbf{x})$ are atomic field operators which create and annihilate atoms at position $\mathbf{x}$, respectively. They are bosonic operators:

$$\left[\hat{\psi}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{x}')\right] = \delta(\mathbf{x} - \mathbf{x}'). \tag{3}$$

This single condensate interpretation is good agreement with most experiments on magnetically trapped alkali atoms. On the other hand, optically trapped condensate has more interesting properties because all spin state are also trapped and nature of condensate will depend on magnetic interaction between different spin states. In other words, the ground state of optically trapped BEC may have more than one macroscopic eigenvalue in its reduced density matrix and called fragmented BEC by interpreting the particles being localized in potential wells. Such fragmented condensate can be regarded as having many different condensate interacting with each other. For spin-1/2 case one may have two component BEC, and for spin-1 case one may have three component BEC.

Double-well or two component BEC.

Recently, great deal of effort has been made on the system consisting two (weakly) interacting BEC. Such a condensate system in principle, can be produced in a double trap with two condensates coupled by quantum tunneling and ground collisions, or in a system with two different magnetic sublevels of an atom, in such case the two species of condensates correspond to the two electronic states involved. For instance, in JILA experiment two condensates were prepared in two different internal atomic states by using a single two-photon coupling pulse and relative phase of the components were measured [4]. One important property of two-mode BEC is that there is not only nonlinear self interaction but also interspecies nonlinear interaction.

Therefore, in the formalism of the second quantization, general form of the Hamiltonian for two component BEC can be written as

$$H = \sum_i \int d\mathbf{x}\hat{\psi}_i^\dagger\left[-\frac{\hbar^2}{2m}\nabla^2 + V_T(\mathbf{x})\right]\hat{\psi}_i(\mathbf{x}) + \frac{g}{2}\sum_{ijkl} U_{ijkl}\int d\mathbf{x}\hat{\psi}_i^\dagger(\mathbf{r})\hat{\psi}_j^\dagger(\mathbf{r})\hat{\psi}_k(\mathbf{x})\hat{\psi}_l(\mathbf{x}) \quad (i,j,k,l = 1,2) \tag{4}$$

where $\hat{\psi}_i^\dagger(\mathbf{x})$ and $\hat{\psi}_i(\mathbf{x})$ are the atomic field operators which create and annihilate atoms of ith sort

($i = 1, 2$) at position $\mathbf{x}$ so that $\left[\hat{\psi}_i(\mathbf{x}), \hat{\psi}_j^\dagger(\mathbf{x}')\right] = \delta_{ij}\delta(\mathbf{x}-\mathbf{x}')$.

Usually this Hamiltonian is reduced to two-mode boson Hamiltonian by expanding the atomic field operators over single-particle states

$$\hat{\psi}_i(\mathbf{x}) = \hat{a}_i\phi_i(\mathbf{x}) + \tilde{\psi}_i(\mathbf{x}), \quad (5)$$

where $\hat{a}_i = \int d\mathbf{x}\phi_i(\mathbf{x})\hat{\psi}_i^\dagger(\mathbf{x})$ creates particles with distributions $\phi_i(\mathbf{x})$ and $[\hat{a}_i, \hat{a}_i^\dagger] = 1$. The first term in mode expansion acts only on condensate state vector, whereas the second term $\tilde{\psi}_i(\mathbf{x})$ accounts for noncondensed atoms. Substituting the mode expansions of atomic field operators in the Hamiltonian, retaining only the first term associated with the condensate, one arrives at the two-mode approximated Hamiltonian:

$$H = \sum \omega_i\hat{a}_i^\dagger\hat{a}_i + \sum_{ijkl} \chi_{ijkl}\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_k\hat{a}_l \quad (i,j,k,l = 1,2) \quad (6)$$

when

$$\omega_i = \int d\mathbf{x}\phi_i^\dagger\left[\frac{p^2}{2m} + V_T(\mathbf{x})\right]\phi_i(\mathbf{x}), \quad (7)$$

$$\chi_{ijkl} = \frac{1}{2}gU_{ijkl}\int d\mathbf{x}\phi_i^*(\mathbf{x})\phi_j^*(\mathbf{x})\phi_k(\mathbf{x})\phi_l(\mathbf{x}). \quad (8)$$

We note that this two-mode approximation involves only first order effects of interaction and valid only for weak nonlinearity. It should be applicable for small number of atoms (for instance, one may estimate $N \le 2000$ with the typical value $a_{scat} = 5$nm for a trap of size 10μm).

Many interesting Hamiltonians can be obtained by designing experiments for various coupling. For instance, Menotti *et. al.* [11] have studied dynamic splitting of BEC by designing the Hamiltonian of the form

$$H = \frac{1}{2}gU_1(\hat{a}_1^{\dagger 2}\hat{a}_1^2 + \hat{a}_2^{\dagger 2}\hat{a}_2^2) - \lambda(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1) + 2gU_2\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_1\hat{a}_2 + \frac{1}{2}gU_2(\hat{a}_1^{\dagger 2}\hat{a}_2^2 + \hat{a}_2^{\dagger 2}\hat{a}_1^2)$$

with $U_1 = \int d\mathbf{x}|\phi_i(\mathbf{x})|^4$, $U_2 = \int d\mathbf{x}|\phi_i(\mathbf{x})|^2|\phi_j(\mathbf{x})|^2$, $U_3 = \int d\mathbf{x}|\phi_i(\mathbf{x})|^2\phi_i^*(\mathbf{x})\phi_j(\mathbf{x})$ $(i,j = 1,2)$, $K_{ij} = \int d\mathbf{x}\phi_i^*(\mathbf{x})\frac{p^2}{2m}\phi_j(\mathbf{x})$, $V_{ij} = \int d\mathbf{x}\phi_i^*(\mathbf{x})V_T(\mathbf{x})\phi_j(\mathbf{x})$ and $\lambda = -K_{12} - V_{12} - gNU_3$. Here it was also used that $\hat{a}_1^{\dagger 2}\hat{a}_1\hat{a}_2 + \hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_2^2 = (N-1)\hat{a}_1^\dagger\hat{a}_2 \approx N\hat{a}_1^\dagger\hat{a}_2$.

By using angular momentum operators defined as

$$S_x = \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1\right)$$
$$S_y = \frac{i}{2}\left(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1\right)$$
$$S_z = \frac{1}{2}\left(\hat{a}_2^\dagger\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_1\right)$$

we can write

$$H = g(U_1 - 2U_2)S_z^2 - 2\lambda S_x + g(S_x^2 - S_y^2).$$

The first term of this Hamiltonian is one-axis twisting model, second term is just rotation, and the last term is nothing but two-axis twisting Hamiltonian.

In a similar way many papers on BEC are actually deals with one-axis [12–14] and two-axis twisting models [11, 15, 16] of spin squeezing. Thus it is important to to find exact solution for two-axis twisting model. We will provide in the next section an exact solution for the problem. In fact we will reduce the problem to the system of linear equations. Having the exact solution we will illustrate some entangled related quantities in some examples.

III. TWO-AXIS TWISTING HAMILTONIAN

A Hamiltonian of two-axis twisting model of spin squeezing for N number of spin-$\frac{1}{2}$ is originally defined as:

$$H_{ta} = \frac{\hbar\chi}{2i}(S_+^2 - S_-^2). \quad (9)$$

In general it is equivalent to any Hamiltonian of the form $H \sim S_i^2 - S_j^2$, where S_i and S_j are any two mutually orthogonal components of the collective angular momentum operators.

The formal solution $|\Psi(t)\rangle = \exp[\alpha(S_+^2 - S_-^2)]|\Psi_0\rangle$ ($\alpha = -\chi t/2$) of the Schrödinger equation may not simply be expressed by a solvable one-axis twisting model, since a simple disentangling, analogous Baker-Campbell-Hausdorff formula for bosonic operators, is not possible. It should be noted that for very large number of N, an approximate solution can be found using Trotter formula

$$\lim_{N\to\infty} e^{i\hat{A}t/N}e^{i\hat{B}t/N} = e^{i(\hat{A}+\hat{B})t}. \quad (10)$$

Since the Hamiltonian Eq. (9) is defined by collective operators, the state always stays in the subspace $\mathcal{H}_J$ with the maximum momentum J and could be written in the form

$$|\Psi(t)\rangle = \sum_{m=-J}^{J} A_m(t)|J,m\rangle \quad (11)$$

as in Eq. (1) discussed in Ref. [9]. We will show here that the coefficients $A_m(t)$ are sum of simple harmonics like $A_m(t) = \sum_j(p_j\cos[\sqrt{\lambda_j}\alpha] + q_j\sin[\sqrt{\lambda_j}\alpha])$, where λ_j is defined by a system of linear equations, and p_j, q_j are defined by the initial state. In other word problem reduces to solving a system of linear equations. General methods of solving system of linear equations may limit the number of equations to be solved. However, we observe that the λ_j has simple (and concavely) distribution as shown in Fig. 1, which may allow one to utilize more specific method of solution. A well structured form of the coefficients of the linear equations hints that an explicit form of λ_j or full analytic solution may exist. Having the exact solution one is able to do entanglement related calculations, which will be illustrated in some examples.

A. Demonstration of exact solution

The idea and mathematics used for providing a general solution are simple and straightforward. But, providing a general formulas for equations and solution requires tedious calculations and the formulas are not easily readable in the general form. So let me apply the approach: from simple to complex.

If the initial state is a CSS, for concreteness let us take it to be $|J,J\rangle$, then evolution of the state always remains in $\mathcal{H}_J$ subspace and may be written in $|J,m\rangle$ basis as

$$|\Psi(t)\rangle = e^{\alpha(S_+^2-S_-^2)}|J,J\rangle = \sum_{k=0}^{\infty}\frac{\alpha^k}{k!}\left(S_+^2-S_-^2\right)^k|J,J\rangle = \sum_{m=-J}^{J}A_m|J,m\rangle \quad (12)$$

and we will find the coefficient A_m.

The term in the first sum of this formula can also be expanded in $|J,m\rangle$ basis

$$(S_+^2-S_-^2)^k|J,J\rangle = \sum_{m=-J}^{J}a_m(k)|J,m\rangle \quad (13)$$

so that

$$|\Psi(t)\rangle = \sum_{k=0}^{\infty}\frac{\alpha^k}{k!}\left(S_+^2-S_-^2\right)^k|J,J\rangle = \sum_{m=-J}^{J}\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}a_m(k)|J,m\rangle = \sum_{m=-J}^{J}A_m|J,m\rangle \quad (14)$$

where we have used the following notation

$$A_m = \sum_{k=0}^{\infty}\frac{\alpha^k}{k!}a_m(k). \quad (15)$$

In mathematics the function of the form as Eq. (15) is known as exponential generating function for a_m. Therefore the problem now shifts to the finding the coefficient a_m and then find the associated generating function A_m.

For simplicity of notation, let's define

$$b_+(m) = \sqrt{(J-1-m)(J-m)(J+1+m)(J+2+m)},$$
$$b_-(m) = \sqrt{(J-1+m)(J+m)(J+1-m)(J+2-m)} \quad (16)$$

using the following equation

$$(S_+^2-S_-^2)|J,m\rangle = b_+|J,m+2\rangle - b_-|J,m-2\rangle. \quad (17)$$

From the definition it is clear that the following properties holds:

$$b_+(m) = b_-(-m), \quad b_+(m) = b_+(-m-2),$$
$$b_-(m) = b_-(-m+2), \quad b_-(m+2) = b_+(m),$$
$$b_+(m-2) = b_-(m). \quad (18)$$

Using them it is easy to see that

$$(S_+^2-S_-^2)^{k+1}|J,J\rangle = \sum_{m=-J}^{J}a_m(k)(S_+^2-S_-^2)|J,m\rangle =$$
$$\sum_{m=-J}^{J}a_m(k)\left(b_+(m)|J,m+2\rangle - b_-(m)|J,m-2\rangle\right) =$$
$$\sum_{m=-J}^{J}[a_{m-2}(k)b_+(m-2) - a_{m+2}(k)b_-(m+2)]|J,m\rangle$$
$$= \sum_{m=-J}^{J}[b_-(m)a_{m-2}(k) - b_+(m)a_{m+2}(k)]|J,m\rangle$$

and equating it with $(S_+^2-S_-^2)^{k+1}|J,J\rangle = \sum_{m=-J}^{J}a_m(k+1)|J,m\rangle$ we get the following system of recurrent equations

$$a_{k+1} = b_-(m)a_{m-2}(k) - b_+(m)a_{m+2}(k)$$
$$(m=-J,-J+2,-J+4\ldots), \quad (19)$$
$$a_m(k) = 0 \quad (m=-J+1,-J+3,\ldots). \quad (20)$$

The latter is an implication of the simple choose of the initial state as $|J,J\rangle$. From this point, let me continue the demonstration for $N=4n$ $(n=1,2,\ldots)$ case just in order to avoid a complicated fractional indices. (Of course, the method and procedure would not be altered for any N.) Then, it is easy to note that always even indices appeared for the coefficient a as a_{2m}. Therefore, for the simplicity of notation we use a new labeling for indices

$$a_{2m} \to a_m \quad \text{with } m=-n,n-1,\ldots n \quad (n=N/4). \quad (21)$$

And by introducing a new notation

$$c_m \equiv b_-(2m) = b_+(-2m) \quad (m=0,1,\ldots,n)$$

the system of recurrent formulas Eq. (19) for the new coefficients $a_m(k)$ can be rewritten as:

$$\begin{cases} a_{-m}(k+1) = c_{m+1}a_{-m-1}(k) - c_m a_{-m}(k) \\ a_0(k+1) = c_1[a_{-1}(k) - a_1(k)] \\ a_m(k+1) = c_m a_{m-1}(k) - c_{m+1}a_{m+1}(k) \end{cases} \quad (22)$$

with $(m = 1,\ldots n)$, and the first element different from zero are given by

$$a_{n-k}(k) = (-1)^{k+1}\prod_{i=1}^{n-k} c_i$$
$$\text{for } k = 1,\ldots,n \quad \text{and} \quad a_n(0) = 1.$$

So far, we have obtained a complicated system of recurrent equation for $a_m(k)$ in straightforward way, and we have to find generalized exponential function for it.

Our trick around here is the introduction of the following new substitution variables

$$f_{\pm m} = a_{-m} \pm a_m, \quad (m = 1,\ldots n) \quad \text{and } f_0 = a_0 \tag{23}$$

then system of equation Eq. (46) splits into two different systems of recurrent equations, i.e,

the first system is:

$$\begin{cases} f_m(k+1) = c_{m+1}f_{-(m+1)}(k) - c_m f_{-(m-1)}(k) \\ \quad (m = 1,3,\ldots) \\ f_{-m}(k+1) = c_{m+1}f_{m+1}(k) - c_m f_{m-1}(k) \\ \quad (m = 2,4,\ldots) \end{cases} \tag{24}$$

and the second system is:

$$\begin{cases} f_m(k+1) = c_{m+1}f_{-(m+1)}(k) - c_m f_{-(m-1)}(k) \\ \quad (m = 2,4,\ldots) \\ f_{-m}(k+1) = c_{m+1}f_{m+1}(k) - c_m f_{m-1}(k) \\ \quad (m = 1,3,\ldots) \\ f_0(k+1) = c_1 f_{-1}(k) \end{cases} \tag{25}$$

with the initial values

$$f_m(n-k-i) = 0 \quad (i = 1,\ldots,n-k),$$
$$f_m(n-k) = (-1)^m \prod_{i=m+1}^{n} c_i \quad (m = 0,\ldots,n), \tag{26}$$

$$f_{-m}(n-k-i) = 0 \quad (i = 1,\ldots,n-k),$$
$$f_{-m}(n-k) = (-1)^{m+1} \prod_{i=m+1}^{n} c_j \quad (m = 1,\ldots,n). \tag{27}$$

Henceforth, we use the index $r = 1,2$ for any variable which is associated with one of the above two separate systems of recurrent equations:

- $r = 1$ refers to the first system of recurrent equations, Eq. (45), by which the functions $f_1, f_{-2}, f_3, f_{-4}, f_5,\ldots$ should be defined,
- $r = 2$ refers to the second system of recurrent equations, Eq. (25), or to the functions f_0, $f_{-1}, f_2, f_{-3}, f_4,\ldots$ are defined.

After heavy but straightforward algebra using substitutions and relabeling it can be shown that each variable of the first system of equations satisfy the same polynomial recurrent formula of the form

$$d_{1,0}f_m(k) + d_{1,1}f_m(k-2) + d_{1,2}f_m(k-4) + \ldots = \sum_{l=0}^{n_1} d_{1,l}f_m(k-2l) = 0$$
$$\text{for } f_m = f_1, f_{-2}, f_3, f_{-4},\ldots \tag{28}$$

each variable of the second system of equations satisfy the same polynomial recurrent formula of the form

$$d_{2,0}f_m(k) + d_{2,1}f_m(k-2) + d_{2,2}f_m(k-4) + \ldots = \sum_{l=0}^{n_2} d_{2,l}F_m(k-2l) = 0$$
$$\text{for } f_m = f_0, f_{-1}, f_2, f_{-3}, f_4,\ldots \tag{29}$$

where

$$n_1 = n_2 = \frac{n}{2} \quad \text{for even } n,$$

$$n_1 = \frac{n-1}{2}, \quad n_2 = \frac{n+1}{2} \quad \text{for odd } n \tag{30}$$

and the coefficients are found as

$$d_{r,0} = 1, \quad d_{r,1} = \sum_{i=2(2-r)}^{n} c_i^2,$$

$$d_{r,2} = \sum_{i_2=3}^{n}\sum_{i_1=2(2-r)}^{i_2-2} c_{i_1}^2 c_{i_2}^2,$$

$$d_{r,3} = \sum_{i_3=5}^{n}\sum_{i_2=3}^{i_3-2}\sum_{i_1=2(2-r)}^{i_2-2} c_{i_1}^2 c_{i_2}^2 c_{i_3}^2 \quad \text{etc.}$$

Here we have used the fact that $c_0 = c_1$ and we note that the construction of the coefficient are the same for both system of equations; the only difference is that the coefficients $d_{1,l}$ involving in recurrent formula for the variables defined by the first system of equations do not contain the constants c_0 and c_1.

A general formula for calculating the coefficients may be written as

$$d_{r,l} = \sum_{i_l=2l-1}^{n}\ \sum_{i_{l-1}=2(l-1)-1}^{i_l-2}\ \sum_{i_{l-2}=2(l-2)-1}^{i_{l-1}-2}\cdots\sum_{i_2=3}^{i_3-2}\times$$
$$\sum_{i_1=2(2-r)}^{i_2-2} c_{i_1}^2 c_{i_2}^2\cdots c_{i_l}^2 \quad (l = 1,2,\ldots,n_r, \quad r = 1,2).$$

We note that in the large N limit the two systems become equivalent having the same coefficient of $d_l^\infty = \frac{N^{5l}}{120^l l!}$ for both $r = 1$ and $r = 2$ case.

So, the problem is now much simplified: a system of mixed recurrent formulas is reduced to only one separated recurrent formula for $f_m(k)$ $(m = -n, -n+1,\ldots,n)$. B y finding its exponential generating function $G(f_m) = \sum_{k=0}^{\infty}\frac{\alpha^k}{k!}f_m(k)$ we should complete the construction of the solution.

The reduction Eq. (28) or (29) is important result since associated exponential generating function of the simple form exists. It can be explicitly checked that the exponential generating function of the recurrent equation can be written in the form

$$G(f_m) = \sum_{k=0}^{\infty} \tfrac{\alpha^k}{k!} f_m(k) = \sum_{l=1}^{n_r} \left[p_{r,m}(l) \left(\cos\sqrt{|\lambda_{r,l}|}\alpha - 1\right) + q_{r,m}(l) \sin\sqrt{|\lambda_{r,l}|}\alpha\right] - f_m(0) \quad (m = -n, -n+1, \ldots, n) \tag{31}$$

where, the coefficients $p_{r,m}$ and $q_{r,m}$ should be defined by a given initial state, and $\lambda_{r,l}$ is a lth solution of the n_rth degree polynomial equation

$$\lambda_r^{n_r} + d_{r,1}\lambda_r^{n_r-1} + d_{r,2}\lambda_r^{n_r-2} + \ldots + d_{r,n_r} = \sum_{l=0}^{n_r} d_{r,n_r-l}\lambda_r^l = 0. \tag{32}$$

We remember again that $r = 1$ or $\lambda_{1,l}$ should be used for $f_1, f_{-2}, f_3, f_{-4}, f_5, \ldots$ and $r = 2$ or $|\lambda_{2,l}|$ should be used for $f_0, f_{-1}, f_2, f_{-3}, f_4, \ldots$.

The cosine and sine functions are appeared in generating function because of the fact that $\lambda_{r,l} < 0$ which follows from $d_{r,n_r-l} > 0$. Square roots are due to the change of the index by twice in Eqs. (28)-(29), the origin of which is the quadratic form of the two-axis twisting Hamiltonian. It should be noted that the above solution method may also be applied to any nonlinear Hamiltonian of the form $H \sim S_+^k - S_-^k$ $(k = 3, 4, \ldots)$. Then we will encounter cubic roots, quartic roots etc. instead of quadratic root as in Eq. (31). Having all $G(f_m)$ we finally could write the solution Eq. (14) with $A_{\pm m} \equiv G(a_{\pm m}) = \frac{1}{2}[G(f_{+m}) \mp G(f_{-m})]$ and $A_0 = G(a_0) = G(f_0)$.

What has been done all above is summarized as: The the problem of solving the Schrödinger equation with nonlinear Hamiltonian of two-axis twisting is reduced to solving only the polynomial equation (32). In other words we have found an exact solution.

Let us finish the demonstration by providing explicit formulas for p_m and q_m. By expanding the sum in Eq. 31 in a series

$$\sum_{l=1}^{n_r} \left[p_{r,l}\left(\cos\sqrt{|\lambda_{r,l}|}\alpha - 1\right) + q_{r,l}\sin\sqrt{|\lambda_{r,l}|}\alpha\right] = = \sum_{k=1}^{\infty} \tfrac{\alpha^{2k}}{(2k)!}\left(\sum_{l=1}^{n_r}(-1)^k p_{r,l}|\lambda_{r,l}|^k\right) + \sum_{k=0}^{\infty} \tfrac{\alpha^{2k+1}}{(2k+1)!}\left(\sum_{l=1}^{n_r}(-1)^k q_{r,l}|\lambda_{r,l}|^{(2k+1)/2}\right) \tag{33}$$

and equating the coefficients of low order terms with coefficients of Eqs. (26) and (27) defined by the initial state one gets the following two independent linear equations for $p_{r,l}$ and $q_{r,l}$:

$$\sum_{l=1}^{n_r} p_{r,l}|\lambda_{r,l}|^k = (-1)^k f_r(2k), \qquad (k = 1, 2, \ldots n_r - 1) \tag{34}$$

with the same $\frac{\alpha^{2k}}{(2k)!}\alpha^{2k}$, and

$$\sum_{l=1}^{n_r} q_{r,l}|\lambda_{r,l}|^{(2k+1)/2} = (-1)^k f_m(2k+1), \quad (k = 0, 1, \ldots n_r - 1) \tag{35}$$

with the same $\frac{\alpha^{2k+1}}{(2k+1)!}\alpha^{2k+1}$.

It is easy to observe that these system equation are just matrix equations with well-known Vandermonde matrix. Therefore using Laplacian development by minors and Vandermonde matrix property, solution of these linear equations can be explicitly written as

$$p_{r,l} = \frac{\sum_{\mu=1}^{n_r} f_m(2\mu)(-1)^{\mu+n_r}\sum'_{i_1>i_2>\ldots}|\lambda_{r,i_1}||\lambda_{r,i_2}|\ldots|\lambda_{r,i_{(n_r-\mu)}}|}{|\lambda_{r,l}|\prod'_{i\neq l}(|\lambda_{r,l}| - |\lambda_{r,i}|)} = \frac{\sum_{\mu=1}^{n_r} f_m(-1)^{\mu+n_r}(2\mu)\Lambda_{n_r-\mu}(\overline{|\lambda_{r,l}|})}{t_l^2\prod'_{i\neq l}(t_l^2 - t_i^2)}, \tag{36}$$

and

$$q_{r,l} = \frac{\sum_{\mu=1}^{n_r} f_m(2\mu-1)(-1)^{n_r}\sum'_{i_1>i_2>\ldots}|\lambda_{r,i_1}||\lambda_{r,i_2}|\ldots|\lambda_{r,i_{(n_r-\mu)}}|}{t_l\prod'_{i\neq l}(t_l^2 - t_i^2)} = \frac{\sum_{\mu=1}^{n_r} f_m(2\mu-1)(-1)^{\mu+n_r}\Lambda_{n_r-\mu}(\overline{|\lambda_{r,l}|})}{\sqrt{|\lambda_{r,l}|}\prod'_{i\neq l}(|\lambda_{r,l}| - |\lambda_{r,i}|)}. \tag{37}$$

where the prime in Σ' and Π' signs means the exclusion the variable with the index of l, in other words, variable $|\lambda_{r,l}|$ is excluded in sums and products. For more readability of the formula we have introduced the following symmetric functions of any set $\{x_n\} \equiv \{x_1, x_2, \ldots, x_n\}$ as

$\Lambda_1 = x_1 + x_2 + \ldots + x_n$
$\Lambda_2 = x_1x_2 + x_1x_3 + \ldots$
$\Lambda_3 = x_1x_2x_3 + x_1x_2x_4 \ldots$
$\vdots$
$\Lambda_n = x_1x_2\cdots x_n$, that is $\Lambda_k = \sum_{\text{perm}} \prod_{i=1}^{k} x_i$ and one defines $\Lambda_k = 0$ for $k > n$ and $k < 0$, and $\Lambda_0 = n$.

The notation $\Lambda_k(\overline{x}_l) \equiv \Lambda_k(x_1, x_2, \ldots, x_{l-1}, x_{l+1}, \ldots, x_n)$ means a symmetric function of the list of $\{x_n\}$ in which l^{th} element is excluded.

B. Alternative Solution

The coefficients $a_m(k)$ in the recurrent Eq. (19) can be considered as the coefficients in a expansion of function $f_m(z)$ in polynomials of z:

$$f_m(z) = \sum_{k=o}^{\infty} a_m(k)z^{-k}, z = re^{i\varphi}. \quad (38)$$

By knowing $f_m(z)$ function, one can find $a_m(k)$ as

$$a_m(k) = \frac{1}{2\pi}\int_0^{2\pi} f_m(z)z^k d\varphi. \quad (39)$$

with

$$dz = re^{i\varphi}id\varphi \to d\varphi = \frac{dz}{iz}, \quad (40)$$

and $A_m(\alpha)$ function as

$$A_m(\alpha) = \sum_{k=o}^{\infty} \frac{\alpha^k a_m(k)}{k!} = \frac{1}{2\pi i}\oint f_m(z)\sum_{k=0}^{\infty}\frac{\alpha^k z^k}{k!}\frac{dz}{z} = \frac{1}{2\pi i}\oint f_m(z)e^{\alpha z}\frac{dz}{z}. \quad (41)$$

In order to find $f_m(z)$ function, let us multiply Eq. (19) by z^{-k} and take a summation by k:

$$z(f_m(z) - a_m(0)) = b_-(m)f_{m-2}(z) - b_+(m)f_{m+2}(z). \quad (42)$$

Bearing in mind that $b_+(m) = b_-(m+2)$, one can find $f(x)$ by solving the following system of linear equations:

$$a_m(0)z = b_-(m+2)f_{m+2}(z) + zf_m(z) - b_-(m)f_{m-2}(z) \quad m = (J, J-1, ..., -J+1, -J). \quad (43)$$

Here $a_m(0)$ with $m = (J, J-1, ..., -J+1, -J)$ are defined by initial state

$$|\psi(0)\rangle = \sum_{m=-J}^{J} a_m(0)|J, m\rangle. \quad (44)$$

The system of linear equations Eq. (43) splits into the following two independent systems of linear equations

$$\begin{array}{l} a_m(0)z = b_-(m+2)f_{m+2}(z) + \\ zf_m(z) - b_-(m)f_{m-2}(z) \quad m = (J, J-2, \ldots), \\ a_m(0)z = b_-(m+2)f_{m+2}(z) + \\ zf_m(z) - b_-(m)f_{m-2}(z) \quad m = (J-1, J-3, \ldots). \end{array} \quad (45)$$

When the initial state is the atomic coherent state $|J, J\rangle$, the second system becomes of the form $f_m(z) = 0 \quad m = (J-1, J-3, \ldots)$ and the first system becomes of the following simple form

$$\begin{cases} z = zf_J(z) - b_-(J)f_{J-2}(z) \\ 0 = b_-(m+2)f_{m+2}(z) + zf_m(z) - b_-(m)f_{m-2}(z) \\ \qquad m = (J-2, J-4, \ldots) \end{cases} \quad (46)$$

Let us demonstrate this alternative solution for concrete examples.

Example 1. In the case of atomic coherent initial state $|\psi(0)\rangle = |1, 1\rangle$ with $N = 2$ or $J = 1$, Eq. (43) becomes of the form

$$\left.\begin{array}{rcl} zf_1(z) - 2f_{-1}(z) &=& z \\ zf_0(z) &=& 0 \\ 2f_1(z) + zf_{-1}(z) &=& 0 \end{array}\right\}, \quad (47)$$

and the roots are

$$f_1(z) = \frac{Z^2}{z^2+4}, f_0(z) = 0, f_{-1}(z) = -\frac{2Z}{z^2+4}.$$

By integrating Eq. (41) one finds the coefficients as

$$\begin{aligned} A_1(\alpha) &= \frac{1}{2\pi i}\oint f_1(z)e^{\alpha z}\frac{dz}{z} \\ &= \frac{1}{2\pi i}\oint \frac{Z^2}{z^2+4}e^{\alpha z}\frac{dz}{z} = \cos(2\alpha), \\ A_0(\alpha) &= \frac{1}{2\pi i}\oint f_0(z)e^{\alpha z}\frac{dz}{z} = \frac{1}{2\pi i}\oint 0\cdot e^{\alpha z}\frac{dz}{z} = 0, \\ A_{-1}(\alpha) &= \frac{1}{2\pi i}\oint f_{-1}(z)e^{\alpha z}\frac{dz}{z} \\ &= -\frac{1}{2\pi i}\oint \frac{2Z}{z^2+4}e^{\alpha z}\frac{dz}{z} = \sin(2\alpha) \end{aligned}$$

and the wave function is found of the form

$$|\psi(t)\rangle = \cos(2\alpha)|1, 1\rangle - \sin(2\alpha)|1, -1\rangle$$

Example 2. With $N = 4$ and $|\psi(0)\rangle = |2, 2\rangle$ one can write Eq. (43) as

$$\left.\begin{array}{rcl} zf_2(z) - 2\sqrt{6}f_0(z) &=& z \\ zf_1(z) - 6f_{-1}(z) &=& 0 \\ 2\sqrt{6}f_2(z) + zf_0(z) - 2\sqrt{6}f_{-2}(z) &=& 0 \\ 6f_{11}(z) + zf_{-1}(z) &=& 0 \\ 2\sqrt{6}f_1(z) + zf{-1}(z) &=& 0 \end{array}\right\}. \quad (48)$$

By finding roots as

$$f_2(z) = \frac{Z^2+24}{z^2+48}, \quad f_1(z) = 0,$$

$$f_0(z) = -\frac{2\sqrt{6}z}{z^2+48}, \quad f_{-1}(z) = 0, \quad f_{-2}(z) = \frac{24}{z^2+48}$$

and by evaluating integrals Eq. (41) as

$$\begin{aligned} A_2(\alpha) &= \frac{1}{2\pi i}\oint f_2(z)e^{\alpha z}\frac{dz}{z} = \frac{1}{2}(1+\cos(\sqrt{48}\alpha)), \\ A_1(\alpha) &= \frac{1}{2\pi i}\oint f_1(z)e^{\alpha z}\frac{dz}{z} = 0, \\ A_0(\alpha) &= \frac{1}{2\pi i}\oint f_0(z)e^{\alpha z}\frac{dz}{z} = -\frac{1}{\sqrt{2}}\sin(\sqrt{48}\alpha), \\ A_{-1}(\alpha) &= \frac{1}{2\pi i}\oint f_{-1}(z)e^{\alpha z}\frac{dz}{z} = 0, \\ A_{-2}(\alpha) &= \frac{1}{2\pi i}\oint f_{-2}(z)e^{\alpha z}\frac{dz}{z} = \frac{1}{2}(1-\cos(\sqrt{48}\alpha)), \end{aligned}$$

one gets the following wave function:

$$\begin{aligned} |\psi(t)\rangle &= \frac{1}{2}(1+\cos(\sqrt{48}\alpha))|2,2\rangle \\ &- \frac{1}{\sqrt{2}}\sin(\sqrt{48}\alpha)|2,0\rangle + \frac{1}{2}(1-\cos(\sqrt{48}\alpha))|2,-2\rangle. \end{aligned}$$

C. Some analytic solutions

Analytic solutions exist for small number of atoms, and let us write down some of them for the purpose of further reference.

$N=2: |\Psi(t)\rangle = \cos(2\alpha)|1,1\rangle - \sin(2\alpha)|1,-1\rangle$

$N=3: |\Psi(t)\rangle = \cos(\sqrt{12}\alpha)|\frac{3}{2},\frac{3}{2}\rangle - \sin(\sqrt{12}\alpha)|\frac{3}{2},-\frac{1}{2}\rangle$

$N=4: |\Psi(t)\rangle = \frac{1}{2}\left[1+\cos(\sqrt{48}\alpha)\right]|2,2\rangle - \frac{1}{\sqrt{2}}\sin(\sqrt{48}\alpha)|2,0\rangle + \frac{1}{2}\left[1-\cos(\sqrt{48}\alpha)\right]|2,-2\rangle$

$N=5$:

$$\begin{aligned} |\Psi(t)\rangle = &\frac{1}{b_{1/2}^2+b_{5/2}^2}\left(b_{1/2}^2 + b_{5/2}^2\cos\sqrt{b_{1/2}^2+b_{5/2}^2}\alpha\right)|5/2,5/2\rangle \\ &-\frac{b_{5/2}}{\sqrt{b_{1/2}^2+b_{5/2}^2}}\sin\sqrt{b_{1/2}^2+b_{5/2}^2}\alpha|5/2,1/2\rangle + \frac{b_{1/2}b_{5/2}}{b_{1/2}^2+b_{5/2}^2}\left(1-\cos\sqrt{b_{1/2}^2+b_{5/2}^2}\alpha\right)|5/2,-3/2\rangle \end{aligned}$$

with $b_{1/2} = 6\sqrt{2}$ and $b_{5/2} = 2\sqrt{10}$.

$N=6$:

$$\begin{aligned} |\Psi(t)\rangle = &\frac{1}{D}\left(K_-\cos K_+^2\alpha + K_+\cos K_-^2\alpha\right)|3,3\rangle - \frac{b_3}{D}\left(K_+^{-1}\sin K_+^2\alpha + K_-^{-1}\sin K_-^2\alpha\right)|3,1\rangle \\ &+\frac{b_3}{D}\left(-\cos K_+^2\alpha + \cos K_-^2\alpha\right)|3,-1\rangle + \frac{b_3^2}{D}\left(K_+^{-2}\sin K_+^2\alpha - K_-^{-2}\sin K_-^2\alpha\right)|3,-3\rangle \end{aligned}$$

with $D = \sqrt{b_1^2+4b_3^2}$, $K_\pm = \frac{1}{2}(D\pm b_1)$, $b_1 = 12$ and $b_3 = 2\sqrt{15}$.

$N=7$:

$$\begin{aligned} |\Psi(t)\rangle = &\frac{b_{-1/2}^2}{R}\left[\left(1-\frac{b_{-1/2}^2}{K_+^2}\right)\cos(K_+\alpha) - \left(1-\frac{b_{-1/2}^2}{K_-^2}\right)\cos(K_-\alpha)\right]|\tfrac{7}{2},\tfrac{7}{2}\rangle \\ &+\frac{b_{7/2}}{R}\left[\left(\frac{b_{-1/2}^2-K_+^2}{K_+}\right)\sin(K_+\alpha) - \left(\frac{b_{-1/2}^2-K_-^2}{K_-}\right)\sin(K_-\alpha)\right]|\tfrac{7}{2},\tfrac{3}{2}\rangle \\ &-\frac{b_{7/2}b_{3/2}}{R}\left[\cos(K_+\alpha)-\cos(K_-\alpha)\right]|\tfrac{7}{2},-\tfrac{1}{2}\rangle + \frac{b_{7/2}b_{3/2}b_{-1/2}}{R}\left[\frac{1}{K_+}\sin(K_+\alpha) - \frac{1}{K_-}\sin(K_-\alpha)\right]|\tfrac{7}{2},-\tfrac{5}{2}\rangle \end{aligned}$$

with $R = \sqrt{(b_{7/2}^2+b_{3/2}^2+b_{-1/2}^2)^2 - 4b_{7/2}^2b_{-1/2}^2}$,

$$K_\pm = \sqrt{\frac{1}{2}\left[b_{7/2}^2+b_{3/2}^2+b_{-1/2}^2 \pm \sqrt{(b_{7/2}^2+b_{3/2}^2+b_{-1/2}^2)^2 - 4b_{7/2}^2b_{-1/2}^2}\right]},$$

$b_{7/2} = 2\sqrt{21}$, $b_{3/2} = 4\sqrt{15}$, and $b_{-1/2} = 6\sqrt{5}$.

$N=8$:

$$\begin{aligned} |\Psi(t)\rangle = &\left[\frac{1}{2}\cos c_2\alpha + \frac{c_2^2}{R^2}(1+\frac{1}{2}\cos R\alpha)\right]|4,4\rangle - \left[\frac{1}{2}\sin c_2\alpha + \frac{c_2}{2R}\sin R\alpha)\right]|4,2\rangle + \frac{c_1c_2}{R^2}(1-\cos R\alpha)|4,0\rangle \\ &+\left[-\frac{1}{2}\sin c_2\alpha + \frac{c_2}{2R}\sin R\alpha)\right]|4,-2\rangle + \left[-\frac{1}{2}\cos c_2\alpha + \frac{c_2^2}{R^2}(1-\frac{1}{2}\cos R\alpha)\right]|4,-4\rangle \end{aligned}$$

with $R = \sqrt{2c_1^2+c_2^2}$, $c_0 = c_1 = 6\sqrt{10}$ and $c_2 = 4\sqrt{7}$.

$N=12$:

$$|\Psi(t)\rangle = A_6|6,6\rangle + A_4|6,4\rangle + A_2|6,2\rangle + A_0|6,0\rangle + A_{-2}|6,-2\rangle + A_{-4}|6,-4\rangle + A_{-6}|6,-6\rangle$$

where

$$A_{\pm 6} = \frac{c_3^2}{2K_2^2} \cos K_2\alpha \pm \frac{c_3^2}{2D_1} \left[\left(1 - \frac{2c_1^2}{K_{+1}^2}\right) \cos K_{+1}\alpha - \left(1 - \frac{2c_1^2}{K_{-1}^2}\right) \cos K_{-1}\alpha \right] + \frac{c_2^2}{2(c_2^2 + c_3^2)} ,$$

$$A_{\pm 4} = \frac{c_3}{2D_1} \left[\left(\frac{2c_1^2}{K_{+1}} - K_{+1} \right) \sin K_{+1}\alpha - \left(\frac{2c_1^2}{K_{-1}} - K_{-1} \right) \sin K_{-1}\alpha \right] \mp \frac{c_3}{2K_2} \sin K_2\alpha ,$$

$$A_{\pm 2} = \pm \frac{c_2 c_3}{2D_1} (\cos K_{-1}\alpha - \cos K_{+1}\alpha) - \frac{c_2 c_3}{2K_2^2} \cos K_2\alpha + \frac{c_2 c_3}{2K_2^2} ,$$

$$A_0 = \frac{c_1 c_2 c_3}{D_1} (K_{+1}^{-1} \sin K_{+1}\alpha - K_{-1}^{-1} \sin K_{-1}\alpha)$$

with $c_0 = c_1 = 4\sqrt{105}$, $c_2 = 6\sqrt{30}$, $c_3 = 2\sqrt{66}$,
$D_1 = \sqrt{(2c_1^2 + c_2^2 + c_3^2)^2 - 8c_1^2 c_3^2}$, $K_{\pm 1} = \sqrt{\frac{1}{2}(2c_1^2 + c_2^2 + c_3^2 \pm D_1)}$, and $K_2 = \sqrt{c_2^2 + c_3^2}$
$N = 16$:

$$\begin{aligned} |\Psi(t)\rangle &= A_8|8,8\rangle + A_6|8,6\rangle + A_4|8,4\rangle + A_2|8,2\rangle + A_0|8,0\rangle \\ &\quad + A_{-2}|8,-2\rangle + A_{-4}|8,-4\rangle + A_{-6}|8,-6\rangle + A_{-8}|8,-8\rangle \end{aligned}$$

where

$$\begin{aligned} A_{\pm 8} &= \tfrac{c_4^2}{2D_1} \left[\left(1 - \tfrac{2c_1^2 + c_2^2}{K_{+1}^2}\right) \cos(K_{+1}\alpha) - \left(1 - \tfrac{2c_1^2 + c_2^2}{K_{-1}^2}\right) \cos(K_{-1}\alpha) \right] \\ &\quad \pm \tfrac{c_4^2}{2D_2} \left[\left(1 - \tfrac{c_1^2}{K_{+2}^2}\right) \cos(K_{+2}\alpha) - \left(1 - \tfrac{c_1^2}{K_{-2}^2}\right) \cos(K_{-2}\alpha) \right] + \tfrac{c_1^2 c_3^2}{c_2^2 c_4^2 + 2c_1^2 (c_3^2 + c_4^2)} \end{aligned}$$

$$\begin{aligned} A_{\pm 6} &= \pm \tfrac{c_4}{2D_1} \left[\left(\tfrac{2c_1^2 + c_2^2}{K_{+1}} - K_{+1} \right) \sin(K_{+1}\alpha) - \left(\tfrac{2c_1^2 + c_2^2}{K_{-1}} - K_{-1} \right) \sin(K_{-1}\alpha) \right] \\ &\quad + \tfrac{c_4}{2D_2} \left[\left(\tfrac{c_2^2}{K_{+2}} - K_{+2} \right) \sin(K_{+2}\alpha) + \left(K_{-2} - \tfrac{c_2^2}{K_{-2}} \right) \sin(K_{-2}\alpha) \right] \end{aligned}$$

$$\begin{aligned} A_{\pm 4} &= \tfrac{c_3 c_4}{2D_1} \left[-\left(1 - \tfrac{2c_1^2}{K_{+1}^2}\right) \cos(K_{+1}\alpha) + \left(1 - \tfrac{2c_1^2}{K_{-1}^2}\right) \cos(K_{-1}\alpha) \right] \\ &\quad \pm \tfrac{c_3 c_4}{2D_2} \left[-\cos(K_{+2}\alpha) + \cos(K_{-2}\alpha) \right] + \tfrac{c_1^2 c_3 c_4}{c_2^2 c_4^2 + 2c_1^2 (c_3^2 + c_4^2)} \end{aligned}$$

$$\begin{aligned} A_{\pm 2} &= \pm \tfrac{c_2 c_3 c_4}{2D_1} \left[K_{+1}^{-1} \sin(K_{+1}\alpha) - K_{-1}^{-1} \sin(K_{-1}\alpha) \right] + \\ &\quad \tfrac{c_2 c_3 c_4}{2D_2} \left[K_{+2}^{-1} \sin(K_{+2}\alpha) - K_{-2}^{-1} \sin(K_{-2}\alpha) \right] \end{aligned}$$

$$A_0 = \tfrac{c_1 c_2 c_3 c_4}{D_1} \left[K_{+1}^{-2} \cos(K_{+1}\alpha) - K_{-1}^{-2} \cos(K_{-1}\alpha) \right] + \tfrac{c_1 c_2 c_3 c_4}{c_2^2 c_4^2 + 2c_1^2 (c_3^2 + c_4^2)}$$

and
$c_0 = c_1 = 12\sqrt{35}$, $c_2 = 6\sqrt{110}$, $c_3 = 2\sqrt{546}$, $c_4 = 4\sqrt{30}$,
$D_1 = \sqrt{(2c_1^2 + c_2^2 + c_3^2 + c_4^2)^2 - 4\left[2c_1^2 (c_3^2 + c_4^2) + c_2^2 c_4^2\right]}$, $D_2 = \sqrt{(c_2^2 + c_3^2 + c_4^2)^2 - 4c_2^2 c_4^2}$,
$K_{\pm 1} = \sqrt{\frac{1}{2}(2c_1^2 + c_2^2 + c_3^2 + c_4^2 \pm D_1)}$, $K_{\pm 2} = \sqrt{\frac{1}{2}(c_2^2 + c_3^2 + c_4^2 \pm D_2)}$.

D. Entanglement and squeezing property

Having analytic solutions we can handle any problem associated with the Hamiltonian. For instance we have calculated the percentual improvement of Ramsey spectroscopy, which is discussed in Ref. [17], with an initial state prepared by two-axis twisting model. The plot is shown in Fig. 6 and we got a little improvement in precision than one-axis twisting because decoherence washes out the details of spin squeezing.

Let us consider entanglement and squeezing property during the evoltion. It can be easily checked that states evolved under two-axis twisting Hamiltonian satisfies the condition of high symmetry except for $N = 3$ case. Therefore, we have the following one to one correspondence between spin squeezing and quantum pairwise entanglement:

$$C_{\text{pair}} = \frac{1 - \xi_{\perp}}{N - 1}. \quad (49)$$

When $N = 3$ we have: $C_{\text{pair}} = \frac{|1-\xi_{\perp}|}{N-1}$.

Since the possible maximum of C_{pair} is $2/N$, we plot concurrence $\frac{N}{2}C_{\text{pair}} \leq 1$ in Figs. (2) -(5).

Pairwise negative correlation or spin squeezing always appears at the beginning short stage of evolution, although the evolution picture becomes more chaotic for large N. This property is familiar to one-axis twisting behavior (compare with Fig. 6 of the Ref. [9]). As we have shown in Ref. [17], such pairwise entanglement improves the frequency measurement precision in the presence of decoherence. For $N \geq 6$ we have quasi-periodic behaviour in entanglement evolution under two-axis twisting Hamiltonian. For large N one may expect chaotic behaviour.

IV. CONCLUSION

BEC is a macroscopic object consisting of genuine indistinguishable atoms. Nonlinear Hamiltonians of BEC should be expressed by collective operators. It could have not only second order nonlinearity as in two-mode approximation but also any high order nonlinearity. Many kinds of nonlinear Hamiltonians for preparing entanglement and squeezing states can be designed. Entanglement and squeezing properties can be studied using the result obtained in Ref. [9] provided that the wave function is found for a given Hamiltonian. But, in general, the later problem is difficult. In this Paper, we have dealt with two-axis twisting Hamiltonian for spin squeezing and have found an exact solution. By using the method given in this chapter we hope many other important nonlinear Hamiltonians could also be solved exactly in a similar way. Research in this direction would be very useful and fruitful since experiments on BEC are advancing at a great speed and many important new applications are appearing. For instance, Jakob Reichel and colleagues at the Ludwig-Maximilians University in Munich have recently created an "atom chip" in which electromagnetic fields control the movement of a condensate hovering above an electronic circuit. Entangled macroscopic states of BEC may become the key building ingredient of future quantum computers, like being entanglement container or supplier.

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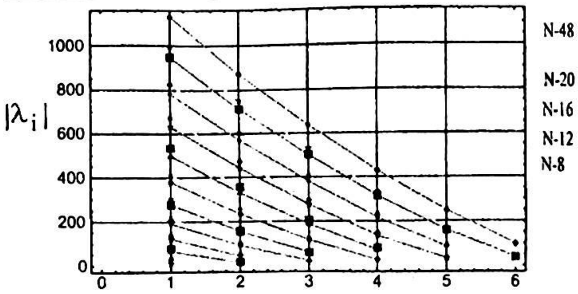

FIG. 1: Root distribution of polynomial equations for different N. The convex and steady property can be used for making the quickest algorithm for solving the polynomial equations. For some coefficients we have explicit expressions.

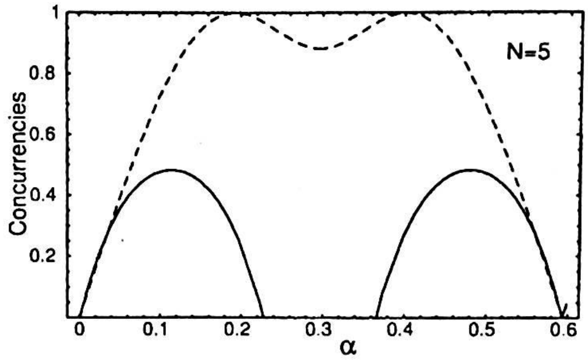

FIG. 4: Evolution of entanglement and squeezing properties of two-axis twisting model for $N = 5$. Solid line is $(N/2)C_{\text{pair}}$; Dashed line is C_{whole}.

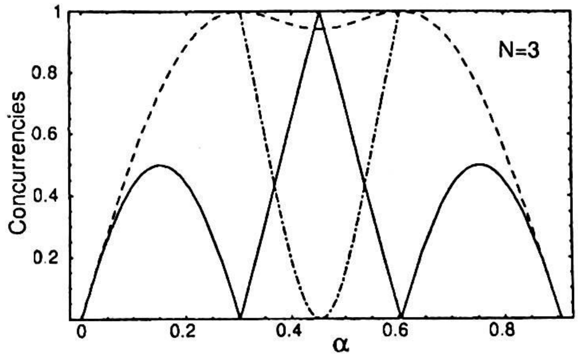

FIG. 2: Evolution of entanglement and squeezing properties of two-axis twisting model for $N = 3$. Solid line is $(N/2)C_{\text{pair}}$; Dashed line is C_{whole}. t is not spin squeezed $\langle S_{\perp}^2\rangle J/2$ in the middle region of $\mu \in [\frac{\pi}{6\sqrt{3}}, \frac{\pi}{3\sqrt{3}}]$. We have also plotted a negative correlation coefficient (Dash-dotted line) $\langle \Delta S_{\vec{n}}^2\rangle(4/J^2)/(1 - \langle S_{\vec{n}}\rangle^2 J^{-2})$.

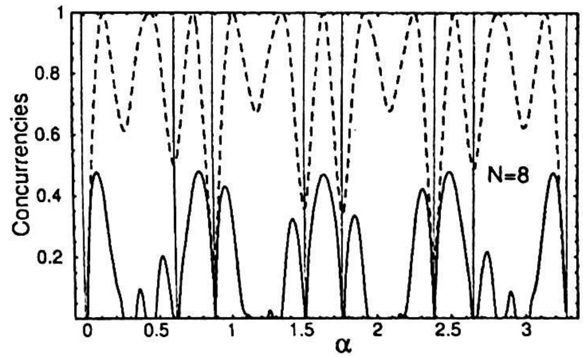

FIG. 5: Evolution of entanglement and squeezing properties of two-axis twisting model for $N = 8$. Solid line is $(N/2)C_{\text{pair}}$; Dashed line is C_{whole}.

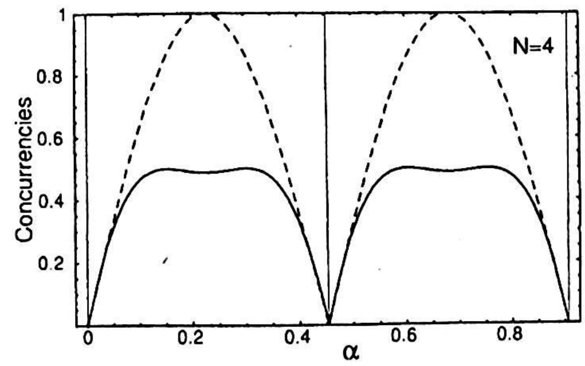

FIG. 3: Evolution of entanglement and squeezing properties of two-axis twisting model for $N = 4$. Solid line is $(N/2)C_{\text{pair}}$; Dashed line is C_{whole}.

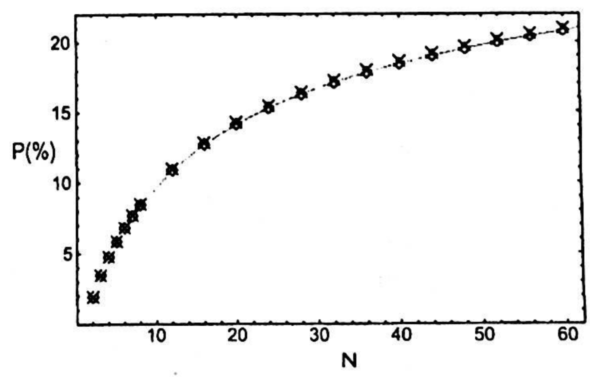

FIG. 6: Two-axis twisting models gives a little percentage improvement than one-axis twisting model because decoherence washes out the details of spin squeezing or entanglement. ×, two-axis twisting; ◇, one-axis twisting