APPLICATION OF STOCHASTIC DIFFERENTIAL EQUATIONS IN POPULATION GROWTH

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Хураангуй: Аливаа улс орны хүн амын тоо нь эдийн засгийн өсөлт, нийгмийн аюулгүй байдал, шийдвэр гаргах үйл ажиллагаа зэрэгт чухал үүрэг гүйцэтгэдэг макроэдийн засгийн үндсэн үзүүлэлт юм. Иймээс хүн амын өсөлтийн динамикийг математикийн янз бүрийн загваруудыг ашиглан урьдчилан таамаглах нь чухал ач холбогдолтой. Хүн амын өсөлтийг голдуу дифференциал тэгшитгэлд үндэслэн хугацаанаас хамаарсан функц хэлбэрээр динамик загвараар тодорхойлдог. Амьдралд хамгийн өргөн хэрэглэгддэг хүн амын загварууд бол бүрэлдэхүүний болоод экспоненциал, логистик загварууд юм. Нөгөө талаас хүн амын өсөлтийг стохастик хувьсагч гэж үздэг. Учир нь хүн амын тоо бол нийгмийн болон улс төрийн бодлого, улс орны эдийн засгийн тогтворгүй байдал зэргээс хамаарч байдаг. Энэхүү судалгааны ажилд эхлээд бид хүн амын уламжлалт загварууд буюу экспоненциал болон логистик загваруудыг авч үзэх болно. Дараа нь хүн амын өсөлтийн харгалзах стохастик загваруудыг судлах ба Монте Карло симуляцын арга ашиглан тэдгээр загваруудыг хооронд нь харьцуулахаас гадна тэдгээрийн тусламжтайгаар Монгол улсын хүн амын 2035 он хүртэлх өсөлтийн динамикийг тооцох болно.

Түлхүүр үгс: Хүн амын загварууд, дифференциал тэгшитгэл, Монте Карло симуляц

Abstract: Population is a key macroeconomic indicator which plays an important role in decision making, social security and economic growth. So it is important to predict the dynamics of population using various mathematical models. Population is described as a function of time usually by dynamic models based on differential equations. The most common practical methods are component, exponential and logistic models. On the other hand, the population growth can be considered as stochastic variables since a number of population depends on social and economic policies, political stability and so on. In this work, we first examine the existing population models such as exponential and logistic model. Then we consider the corresponding stochastic models for population growth, compare these methods using Monte Carlo simulation and predict the dynamics of Mongolian population up to 2035 year by the methods.

Key words: Population models, differential equations, Monte Carlo simulation

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Introduction

Population as a function of time is usually described by dynamic models based on differential equations. The most common practical methods are component, exponential and logistic models. On the other hand, the population growth can be considered as stochastic variables since a number of population depends on social and economic policies, political stability, epidemic decease, plague, disaster and so on. Due to the unpredictable nature of population growth, we introduce uncertainty in the population models. We choose the linear multiplicative noise, that is a random element to the exponential and logistic growth equations.

We applied the deterministic and stochastic models to the Mongolian population growth. In recent years, Mongolian population has grown rapidly and reached 3.1 million people in 2016.

According to the projection of Mongolian Statistical Office, an estimation based on resident population within the territory of Mongolia of United Nations Population Fund shows that the population of Mongolia will reach 4 million around 2038 under the scenario with low fertility decline and 3.5 million around 2030 under the scenario with high fertility decline.

In this paper, we first examine the existing population models such as exponential and logistic model. Then we consider the stochastic models for population growth, compare the models using Monte Carlo simulation and predict the dynamics of Mongolian population up to 2035 year using the models.

Population model

Deterministic models

So far different mathematical models have been developed to model population dynamics. In this section, we introduce two common deterministic population models such as exponential and logistic.

Exponential model

One of the models for population growth is based on the assumption that the population grows at a rate proportional to the number of population. This yields the following initial value problem (IVP)

$$\begin{cases} \frac{dY}{dt} = rY, \\ Y(0) = Y_0. \end{cases} \qquad (1)$$

where r is a measure of the growth rate, t is a period, Y is equal to the number of individuals at time t and Y_0 is the population for the base year. The equation is called the exponential growth. The solution of the initial value problem is $Y(t)=Y_0e^{rt}$.

Logistic model

A population often increases exponentially in its early stages but levels off eventually and approaches its carrying capacity because of limited resources. One of the first to model this limitation was the logistic growth model

$$\begin{cases} \frac{dy}{dt} = ry(1-\frac{y}{M}), \\ y(0) = y_0 \end{cases} \text{ or } \begin{cases} \frac{dy}{dt} = ky(M-y), \\ y(0) = y_0. \end{cases} \quad (2)$$

where M is the carrying capacity or the maximum number of population for a given period.

By solving the initial value problem, we get

$$Y(t) = \frac{MY_0}{Y_0 + (M - Y_0)e^{-Mkt}}.$$

Stochastic models

Before we consider Stochastic models, we introduce the **stochastic process** or the Wiener process and the **stochastic differential equations**.

A stochastic process is a family of random variables that depend on time. The stochastic process can either be a "discrete-time" process or a "continues-time" process. An example of the latter one is Wiener process or Brownian motion.

The Wiener process, $W(t),\ t \geq 0,$ satisfies the following properties.

(i) [Independence of increments] $W(t) - W(s),\ s < t,$ is independent of the past.

(ii) [Normal increment] $W(t) - W(s)$ is normally distributed with mean $\mu = 0$ and variance $\sigma^2 = t - s$.

(iii) [Continuity of paths] $W(t)$ is continuous function of t.

(iv) $W(0) = 0$.

An implication of (ii) and (iv) is that $W(t) \sim N(0, \sqrt{t})$.

By introducing noise into an ordinary differential equation (ODE) a stochastic differential equation (SDE) is obtained. Let $Y(t)$ be an unknown stochastic process, $W(t)$ a Wiener process, $\mu(x,t)$ and $\sigma(x,t)$ known functions, then

$$dY(t) = \mu(Y(t),t)dt + \sigma(Y(t),t)dW(t) \quad (3)$$

is an SDE driven by a Wiener process. The function $\mu(x,t)$ is called the drift coefficient and $\sigma(x,t)$ is called the diffusion coefficient.

An initial value problem for SDE is

$$\begin{cases} dY(t) = \mu(Y(t),t)dt + \sigma(Y(t),t)dW(t), \\ Y(0) = Y_0. \end{cases} \quad (4)$$

As the case of deterministic differential equations, SDEs may not be explicitly solvable. When this is case, numerical methods can be used to obtain an approximation of the solution if a solution exists. The noise is called additive if the diffusion coefficient, σ, doesn't depend on the stochastic process, $Y(t)$, or multiplicative if it does depend on $Y(t)$.

The Stochastic exponential growth model

For simplicity, we consider that the drift coefficient is a proportional to the stochastic process and the diffusion coefficients is a linear multiplicative. Then we obtain the stochastic exponential growth model

$$dY(t) = rY(t)dt + \sigma Y(t)dW \quad (5)$$

where r is the growth rate, σ is number called diffusion coefficient and dW is a Wiener process.

For simulation, let the number of population for base year be y_0 at time t=0 and $t_i = i\Delta t$, so the numbers of population are to be determined at discrete points t_i. Then our discrete-time model is

$$Y(t_{i+1}) = Y(t_i) + r\Delta t Y(t_i) + \sigma\sqrt{\Delta t}X_i Y(t_i),$$

where X_i (i=0,1,2,...) are i.i.d $N(0,1)$.
Based on the Central Limit theorem and X_i (i=0,1,2,...) are i.i.d N(0,1), we take a limit $\Delta t \to 0$ to get continuous population model [1]. Continuous time expression for the model is

$$Y(t) = Y_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma\sqrt{t}X},$$

where $X \sim N(0,\ 1)$.
A random variable $Y(t)$ of the form has a so-called *lognormal distribution*, that is, its log is normally distributed. So we can describe the evolution of the population over any sequence of time points $0 = t_0 < t_1 < t_2 < \cdots < t_M$ by

$$Y(t_{i+1}) = Y(t_i)e^{\left(r-\frac{1}{2}\sigma^2\right)(t_{i+1}-t_i)+\sigma\sqrt{t_{i+1}-t_i}X_i}, \text{ for i.i.d } X_i \sim N(0,1).$$

The Stochastic logistic growth model

We choose the linear multiplicative noise, $\sigma Y(t)$, a random element to the logistic growth equation

$$dY(t) = rY(t)\left(1 - \frac{Y(t)}{M}\right)dt + \sigma Y(t)dW \quad (6)$$

where r is a measure of the growth rate and M is a carrying capacity of the population which limits the growth of the population. The solution can be shown to be ([Kloeden and Platen 1999, p.125])

$$Y(t) = \frac{Y_0 K e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W(t)}}{K + Y_0 r \int_0^t e^{\left(r - \frac{\sigma^2}{2}\right)s + \sigma W(t)} ds} \quad (7)$$

If there is no noise in the system, i.e. when $\sigma = 0$, (7) reduces to the solution of the deterministic logistic growth model. The integral in the denominator of equation (7) is not readily obtainable. However, expected value of it can be calculated and found to be [4]

$$E\big(I(t)\big) = \frac{e^{rt}}{r}, \text{ where } I(t) = \int_0^t e^{\left(r - \frac{\sigma^2}{2}\right)s + \sigma W(s)} ds.$$

Since equation (7) is a quotient of two dependent random variables its expected value is not readily obtainable. However, applying the Delta method the expected value can be approximated by the expected value of the numerator divided by the expected value of the denominator, i.e.

$$E\left(\frac{Y}{Z}\right) \approx \frac{E(Y)}{E(Z)}.$$

So this yields

$$E\big(Y(t)\big) = \frac{Y_0 K e^{rt}}{K + Y_0(e^{rt} - 1)}$$

which is the solution of the deterministic logistic growth model. Thus, the expected value of the solution to the stochastic logistic growth model can be approximated with the solution of the deterministic model.

For discretization, let the number of population for base year be Y_0 at time $t = 0$ and $t_i = i\delta t$, so the numbers of population are to be determined at discrete points t_i. Then our discrete-time model for logistic growth is

$$Y(t_{i+1}) = Y(t_i) + rY(t_i)\left(1 - \frac{Y(t_i)}{M}\right)\delta t + \sigma\sqrt{\delta t}X_i Y(t_i), \quad (8)$$

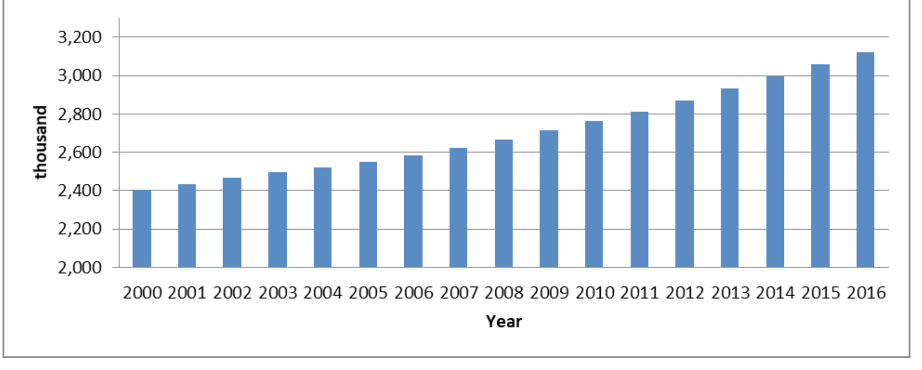
where X_i $(i = 0,1,2,\dots)$ are i.i.d $N(0,1)$.

Simulation results

We applied the deterministic and stochastic models to the Mongolian population growth. In recent years, Mongolian population has grown rapidly and reached 3.1 million people in 2016. In our work, we estimated the growth rate using statistical data from 2000 to 2016 obtained by the National Statistical Office of Mongolia. According to the least square estimation, the average population growth rate was 1.59 during that periods.

We use Monte Carlo simulation to generate the stochastic processes. Since we have already estimated population growth rate as $r = 0.0159$ and and different σ, we can generate sample paths starting from 2403.1 thousands in 2000 using different pseudo random number generators for each path.

Figure 1. Mongolian population, thousand person



Source: National Statistical Office of Mongolia

Figure 2. Stochastic exponential model

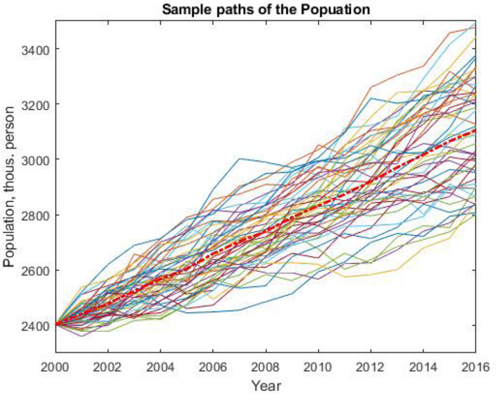
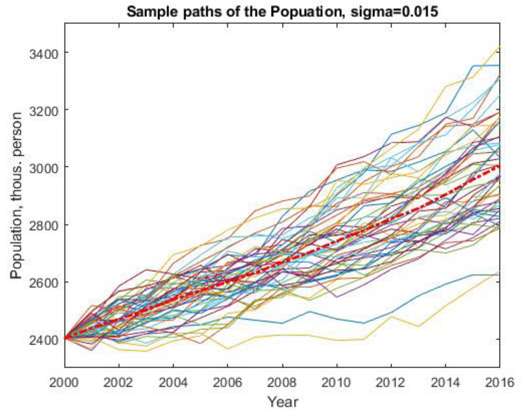


Figure 3. Stochastic logistic model



We compared our models such us deterministic and stochastic models with real data from 2000 to 2016 and predicted Mongolian population during 2017-2035. Results are shown on the following graph and tables. From the graph and table, we can see that stochastic models are more suitable for prediction of the population. According to the stochastic exponential prediction, our population will reach 3309.8 thousand in 2021, 3524.3 in 2025, 3880.3 in 2031 and 4121.1 thousand in 2035 where as the number of population will be 3226.5 thousand in 2021, 3418.6 in 2025 and 3946.8 thousand in 2035 by the stochastic logistic model.

Figure 4. Comparison of the models

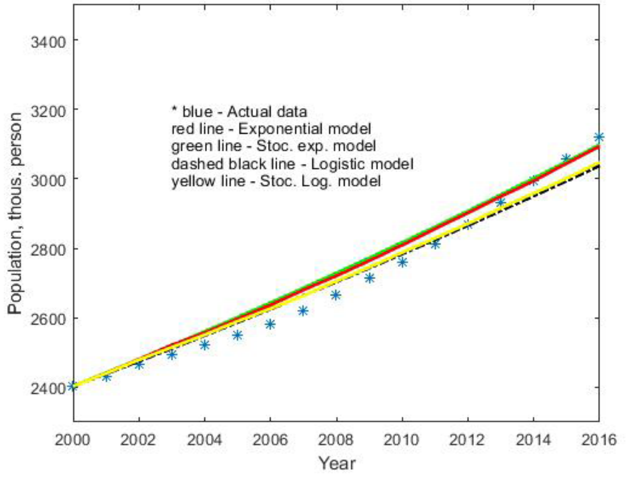


Table 1. Mongolian population (thousand)

Year	Population	Exponential model	Stoc. exp. model	Logistic model	Stoc. log. model
2000	2403.1	2403.1	2403.1	2403.1	2403.1
2001	2432.4	2441.6	2440.8	2438.9	2440.0
2002	2465.7	2480.7	2480.6	2475.2	2479.1
2003	2495.1	2520.4	2521.5	2511.9	2515.0
2004	2521.7	2560.7	2557.0	2549.2	2551.0
2005	2552.1	2601.7	2596.7	2587.0	2589.9
2006	2583.3	2643.3	2636.9	2625.3	2626.6
2007	2620.4	2685.7	2680.3	2664.2	2663.7
2008	2666.0	2728.7	2720.8	2703.5	2705.5
2009	2716.3	2772.3	2764.4	2743.4	2745.6
2010	2761.0	2816.7	2809.4	2783.8	2788.0
2011	2811.7	2861.8	2856.2	2824.8	2829.4
2012	2867.7	2907.6	2902.3	2866.3	2871.3
2013	2930.3	2954.2	2949.2	2908.4	2916.0
2014	2995.9	3001.4	2994.4	2951.0	2958.6
2015	3057.8	3049.5	3044.8	2.9942	3001.5
2016	3119.9	3098.3	3093.4	3038.0	3047.9
Least square errors		2.829e+04	2.087e+04	2.223e+04	1.860e+04

Table 2. Population up to the year 2035 (thousand)

Year	Exponential model	Stoc. exp. model	Logistic model	Stoc. log. model
2017	3098.3	3097.7	3038.0	3041.6
2019	3198.3	3204.5	3127.2	3131.5
2021	3301.5	3309.8	3218.9	3226.5
2023	3408.1	3413.8	3312.9	3320.7
2025	3518.0	3524.3	3409.4	3418.6
2027	3631.6	3638.3	3508.4	3520.9
2029	3748.8	3761.6	3600.0	3623.2
2031	3869.7	3880.3	3714.1	3730.3
2033	3994.6	3999.7	3820.9	3835.5
2035	4123.5	4121.1	3930.3	3946.8

Conclusion

- In this work we only considered the linear diffusion coefficient in exponential and logistic growth models. Other diffusion coefficients are plausible. For example, the logistic growth model can be perturbed with white noise.
- According to statistical data, stochastic models are more suitable for prediction of the population.
- According to our prediction, the Mongolian population will reach 3309.8 thousand in 2021, 3524.3 in 2025, 3880.3 in 2031 and 4121.1 thousand in 2035. Particularly, the population of Mongolia will reach to 4 million around 2033 and 2034. However, the estimation is higher than the projection of Mongolian Statistical Office which estimated that Mongolian population will reach to 4 million around 2038 under scenario only with low fertility decline.

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