MEASUREMENT SCALES AND OPTIMIZATION IN ECONOMICS

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Abstract: This paper explores the impact on the outcome of optimizing a function when permissible transformations of the scale of measurement are applied to the variable with respect to which that function is optimized. The collection of permissible transformations vary depending on whether the variable is measured on a ratio, cardinal (interval), or ordinal scale. Certain problems arise when the variable in question is measured only on an ordinal scale.

Key words: optimization, permissible transformations, scale of measurement, cardinal scale.

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** This paper is an exploration of the mathematics of an issued raised in Katzner [6].

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Analysis often requires the use of numbers that may be understood as being drawn from a measurement scale. This is true, for example, with the statement that distance equals velocity times time. In that equation, distance is measured by length taken from a scale such as miles; velocity from a scale defined by, say, miles per hour; and time from a scale that counts instants or periods. Those three scales are generally characterized as ratio scales. In Economics, ratio scales along with two other kinds of scales, ordinal and cardinal (or interval) scales, have become important. These scales may be characterized as follows:

The current mathematical notion of ordinality dates to the 1950s (Narens.L., 1985). It is based on the idea that ordinal numbers emerge as values that are are located in the image of some order-preserving function (ordinal scale). That function is defined on an abstract space, call it D, on which a reflexive, transitive, and total (complete) relation ordering its elements according to a specific property is imposed, and whose property-ordering is preserved in the function's values. Were D with such an ordering to be specified independently, it can be shown that such an order-preserving function exists if and only if the interval topology for the equivalence classes under the ordering has a countable base (Pfanzagl.J., 1971). The order-preserving function may be said to be a representation of the underlying ordering. And any increasing transformation of the numbers in the range of the function also represents that ordering or, in other words, is a different ordinal scale that preserves the original ordering in the transformed function values. Without additions to this framework, it is not legitimate to perform arithmetic operations with ordinal numbers because there is no underlying basis for doing so. Thus the derivatives of a function whose range consists of ordinal numbers cannot be calculated since their derivations require the use of subtraction and division.

The only way to introduce the possibility of arithmetic manipulation of ordinal numbers is to add to the ordering structure built on the underlying space. This may be accomplished by introducing an additive composition operation[1] on the latter along with the requirements that D is connected and contains at least two elements. The added structure turns the ordinal scale into a cardinal scale (Pfanzagl.J., 1971). The only increasing transformations of scale that can be applied while maintaining on D both the ordering and compositional constructions in a cardinal representation are linear. Ratio scales are cardinal scales having the further feature that the position of the zero does not change under all admissible transformations of scale.[2] In the

[1] A composition operations is additive if it is associative, commutative, cancellable, and continuous in each variable separately.

[2] For example, temperature is a measured on a cardinal, but not ratio scale because the zero changes in moving from, say, Fahrenheit to centigrade. Weight is measured on a ratio scale since the zero is the same regardless of whether measurement is in terms of pounds or kilograms.

case of ratio scales, only dilation transformations of scale are admissible.[3]

These concepts may be illustrated in terms of pieces of chalk. One of the properties of chalk is that each piece has "longness" associated with it. That longness orders the pieces of chalk in a reflexive, transitive and total way and is measured by the order-preserving function length in, say, inches. The length scale is ordinal since a piece of chalk with more longness has greater length associated with it. Also associated with longness is an additive composition operation identified on D as the placing of individual pieces end to end. Thus the longness of the combination of any two pieces of chalk is measured as the sum of the lengths of the individual pieces in it. That is, the length scale is cardinal. It is also a ratio scale since transforming the scale into, say, centimeters leaves the position of the zero unchanged. Such are the basic ideas characterizing the current mathematical notions of ordinal, cardinal, and ratio scales.

However, there is an older concept of ordinality that is defined similarly to that described above except that it permits arithmetic operations with ordinal numbers while ignoring the underlying compositional requirements necessary, in the 1950s approach, to perform them. From this alternative perspective ordinal numbers are also seen as located in the range of an order-preserving function and may be added, subtracted, multiplied, and divided irrespective of any composition operation that might be defined to justify those manipulations. All increasing transformation of scale (not only the linear ones) can be applied to ordinal numbers without disrupting the underlying ordering on D and without losing the ability to perform arithmetic operations on the transformed numbers. In this context, any function whose values are ordinal can be differentiated provided it has the right smoothness properties. Such an 'old fashion' notion of ordinality persisted before it was replaced by the more modern approach of the 1950s (Stevens.S.S, Friday, June 7, 1946).

Now, explicitly or implicitly, old fashion ordinality entered the economics theoretical literature long before the 1950s. As far back as 1892, Fisher [2, pp. 31-33] realized that applying increasing transformations to the ordinal values of a function with a maximum value does not change the element in the domain over which the maximum occurs. Both Hicks [4, pp. 306-307] and Samuelson [11, p. 94] base their respective 1939 and 1958 discussions of the theory of consumer demand on twice, continuously differentiable, (old fashion) ordinal utility functions (i.e., functions with old fashion ordinal function values and domains consisting of vectors of ratio measured quantities of commodities) and the method of Lagrange multipliers. And this old fashion approach to ordinality survives in the Economics literature to this day (Donald W. Katzner, May 2014). The more

3 The linear function $t(x) = ax + b$ where a and b are constant is called a dilation when $b = 0$.

modern approach of the 1950s does not seem to have caught on in any general way among economists. In the remainder of this paper, "ordinal scale" will refer to the old fashion economist notion while "cardinal" and "ration scales" will denote the mathematical conceptualizations as described above.

In many analyses, the choice of which scales to use (*i.e.*, whether to employ, say, inches or centimeters to measure length) is arbitrary. And in most of these the selections made do not matter. This is certainly true of the ordinal utility functions appearing in the theory of consumer demand mentioned above. But, as will be seen, the lack of consequences does not hold up everywhere. In particular, the choice of measurement scales does make an important difference when a function whose argument can be measured only on an ordinal scale is optimized. It should be noted that while the term 'optimization' covers both maximization and minimization with and without constraints, the following focuses only on unconstrained versions of one or the other as indicated by the context of the issue under discussion.

To see what is involved, it is necessary to consider in detail how changes in scale affect the outcome of the optimization process. This is the subject matter of the next two sections. Section I provides a concrete example of how changes in scale of a ratio measured variable with respect to which a function is minimized affects the result of the minimization. In this illustration, the scale change requires a corresponding adjustment of the function to be minimized in order to obtain a consistent outcome. Section II considers the problem more generally. It is here that the difficulty with respect to ordinality is discussed. Although most of that discussion is stated in terms of ordinal scales, much of it also applies when cardinal or ratio scales are in use as long as the restrictions on the nature of admissible scale changes (linear in the case of cardinal scales and dilations for ratio scales) are respected. Section III considers two ways around the ordinality problem raised in Section II. And Section IV provides an example in economics in which, if a variable with respect to which a function is to be maximized is taken to be ordinal (and a case can be made for that ordinality), then the analysis based on that maximization runs into difficulty.

I

Suppose 4 there are two ways to ride a bicycle between points A and D. One way is to follow a straight road from A to point B, and then a perpendicular straight road to D (see Figure 1). The alternative is to cut across a field from A and pursue a straight line to a point C on the road between B and D, and then continue on the latter road to D. The distance between A and B is 8 miles, and that between B and D is 16 miles. Assume the cyclist's average speed on roads is 10 miles per

[4] This example is based on A. Svirin [13, Example 18].

hour, while that across the field is only 6 miles per hour. Suppose the cyclist wants to choose C, which can be located anywhere between B and D, to minimize the riding time between A and B.

In this example, longness is measured by distance in miles. With $x \geq 0$ representing the distance between B and C, the distance between A and C, obtained from the Pythagorean Theorem is $\sqrt{64 + x^2}$. Using the formula time equals distance divided by velocity, the time required to bicycle form A to B is the sum of the times needed from A to C and C to D:

$$T(x) = \frac{\sqrt{64 + x^2}}{6} + \frac{16 - x}{10}.$$

Differentiating with respect to x,

$$T'(x) = \frac{10x - 6\sqrt{64 + x^2}}{60\sqrt{64 + x^2}}$$

and equating that derivative to zero gives the time-minimizing distance[5] from B to C as $x = 6$. The minimum time is $T(6) = 2\frac{2}{3}$. Denote the underlying longness between B to C which is measured as 6 miles by ℓ'.

Now let the scale on which longness is measured be changed by dilation from miles to half-miles. Then x denotes the distance between B and C in modified units (half-miles), the distance between A and B becomes 16, that between B and D expands to 32, and that between A and C is written as $\sqrt{256 + x^2}$. The time function becomes

$$\tilde{T}(x) = \frac{\sqrt{256 + x^2}}{12} + \frac{32 - x}{20},$$

and upon setting the derivative to zero, $x = 12$ and the minimum time remains unchanged at $\tilde{T}(12) = 2\frac{2}{3}$. Represent the underlying longness between B and C in the context of the scale change by ℓ''. Since the transformed value of ℓ' originally measured as $x = 6$ under the scale change to half-mile units is $x = 12$, it is clear that $\ell' = \ell''$. Thus the change in scale does not alter the outcome of the minimization problem. This happens because changing the scale on which longness is measured is accompanied by corresponding modifications in the parameters of T that adjust that function to express the relation that T represents with respect to the new measure. Obviously, T has to be characterized in sufficiently specific form to be able to make the necessary adjustment. But if longness were measured only on an ordinal scale and if T were not defined with enough precision, such modification might not be possible, and it then would not necessarily follow that

[5] It is easily verified that the second-order derivative $T''(x) > 0$ for all $x \geq 0$.

$\ell' = \ell''$. And even if T were specified in sufficient detail, it is still possible, as will be seen subsequently, that applying certain transformations of scale, a value of x that minimizes T would not exist and ℓ'' could not then be determined. The matter will be taken up in the next section in the context of maximization.

II

To be concrete, let e be the variable "effort" ranging over some non-denumerably infinite space E which contains distinct non-numerical, written paragraphs describing the various forms of "effort" that might arise in a particular study.[6] The question of which scale (ordinal, cardinal, or ratio) is appropriate to use in measuring effort is not considered here. The consequences of using one over the other will be discussed in due course. Let there be a reflexive, transitive, and total ordering relation defined on E according to the "strength of effort" thought to arise in each description (variable value). Because they are not quantified, no arithmetic operations can be applied to manipulate the values of e. To overcome this, let there be an order-preserving, one-to-one function $x = g(e)$ mapping E onto the real line such that arithmetic operations may be performed on the values of x. (The fact that x can take on negative numerical values is of no consequence.) The function g is taken to be an ordinal scale or measure of effort that contains no more information than that present in the ordering on E, and x represents ordinally measured effort "strengths." Clearly g is not unique and any strictly increasing transformation t of g such as $z = t(x) = t(g(e))$, where the derivative $t'(x) > 0$ for all x, is another ordinal scale for measuring effort. Both $g(e)$ and $t(g(e))$ represent the same underlying ordering on E, and x and z are equally legitimate measures that can be used to represent the various strengths of different kinds of effort. From a logical point of view, it does not matter which representation is used to record the effort ordering; the choice of one over the other is arbitrary. Substitution of one for another leaves the informational content of the ordering unchanged.

It is not necessary to repeat the details of requiring effort to be measured on a cardinal or ratio scale. Suffice it to say that the choice of the particular scale employed is arbitrary, although the permissible strictly increasing transformations of scale are limited to those that are, respectively, linear or dilations. In all cases the ordering and arithmetic-operation informational content of the measured variable values does not change with variations in the admissible scales. It follows that, in general, the analytical techniques employed to manipulate the numbers on these scales should give the same result regardless of the admissible scale on which

[6] The specification of non-denumerable infinite spaces of non-numerical variable values is not considered here. See Katzner [5, Sect. 7.1].

effort is measured. This is true, often with qualification (as described in terms of modification of the function T in the example of Section I) when the function values of g are cardinal or ratio measured. However, with respect to optimization that is not always the case when those function values are only ordinally measured.

An example employing optimization in which substitution of ordinal scales without any qualification at all does not make a difference occurs, as previously suggested, in the classical theory of consumer demand. In that case, utility may be understood as an ordinal measure of the "pleasure" a consumer would obtain from the various baskets containing quantities of goods (the quantities being ratio measured) he or she might purchase or demand, and is expressed as a function of those baskets. To determine the consumer's demand for baskets, utility is maximized subject to the individual's budget constraint. Applying any increasing transformation to the elements of the range of the ordinal utility function does not change the outcome of the maximization and hence the basket of goods demanded. In this case, ordinality does not affect the outcome of the maximization because it arises with respect to the values of the function being maximized. But, as intimated earlier, when the ordinality appears in terms of an argument of the function with respect to which the maximization is taken, the choice of scale does make an important difference. In particular, the structure of the analysis can break down and in that case the questions being addressed by the optimization cannot be given meaningful answers.

Although the following focuses on the problem of maximizing a function of a single variable, much of the argument is easily generalized.

Consider the above context in which effort e is an ordinally measured variable and let g be chosen to represent the ordering on E. Let f be a function defined on the range of g into the real line indicating the output y obtained from each effort e in E as represented by x. Thus

$$y = f(x) \qquad (1)$$

on $(-\infty \leq x \leq \infty)$. Suppose a goal of the analysis is to find the unique value of e that maximizes output. Suppose also that that value is to be found by first obtaining the unique value x^0 that maximizes the output function f. Then x^0 is to be related back to e through g (recall that g is 1-1). Thus, with enough differentiability it becomes sufficient to determine the x^0 for which $f'(x^0) = 0$ and $f''(x^0) < 0$. The following propositions and discussion provide insight into this approach.

Proposition 1. Let D be a subset of the real line R. Let f map D into R, and t map D onto D. Then f has a maximum at x^0 in D if and only if the composite function $f \cdot t(x) = f(t(x))$ has a maximum at some $\bar{x}$ in D.

Proof: Substituting $t(x)$ for x in f changes the values that f assigns to each x in D. But it does not change the collection of function values of f. The application of t only rearranges the assignment of the function values of f to the elements of its domain. So there is a maximum value among the function values of f if and only if there is one among the function values of $f*t$.

QED

For some functions f and transformations t, it turns out that $\bar{x} = t(x^0)$. This would be the case, for example, if

$$f(x) = -x^2, \qquad (2)$$

and $t(x) = ax$ for any constant $a \neq 0$.[7] For then, both (2) and the composite function $f \cdot t(x) = -(ax)^2$ have maxima at $x^0 = \bar{x} = 0$ and $t(0) = 0$. But there are transformations such as

$t(x) = ax + b$, with b a non-zero constant, where this relationship does not hold. In the latter case, replacing x by $t(x)$ in (2), $f \cdot t(x) = -(ax + b)^2$ which has a maximum at $\bar{x} = -{}^{b}/_{a}$. To compare the underlying value of e from the maximization of f with that from the maximization of $f \cdot t$ when $t(x) = ax + b$, let e^0 be associated with x^0 under g and $\bar{e}$ with $\bar{x}$ under $t \cdot g$ (remember that g and, in this case t, are 1-1). Since x^0 is transformed into $ax^0 + b$ by t, both x^0 in D and $ax^0 + b$ in the transformed space correspond to the same value e^0. But $x^0 \neq \bar{x}$ in D and in the transformed space $ax^0 + b \neq \bar{x}$. Therefore $e^0 \neq \bar{e}$. Thus the outcome of the maximization of $f \cdot t$ depends on the particular scale that is used.

If, instead of representing effort, e were to denote, say, hotness or longness, and measured as x by, respectively, (cardinally measured) temperature or (ratio measured) length, and if output y and the function f were replaced by an appropriate variable and function, then equation (1) can be adjusted, as in the example of Section I, to compensate for the change in scale. That is, changing the measurement of a variable from Fahrenheit to centigrade or from feet to meters can be accompanied by an appropriate adjustment in (1) that allows for retention of the same relation (albeit expressed in different units) in the new statement $y = f \cdot t(x)$, where t is the appropriate increasing transformation (linear or dilation) that is applied to make the switch. Under such conditions, the same outcome e^0 obtains with either scale. But if effort is measured only ordinally, and if there is no generally accepted scale for measuring it and no generally acceptable form for f in (1), then

[7] A generalization of which this this example is a special case appears in Proposition 3 below.

there is no way to identify an acceptable concrete structure for (1). And, since the choices of g, and t are all arbitrary, the question of which value of e maximizes output cannot be answered.

However, even if there were a generally accepted scale for e and form for f, as long as e is only ordinally measured, there is another difficulty. Return to the function of (2) and suppose the value of e that uniquely maximizes output is still to be determined by finding the x^0 that maximizes (2). Of course, in this case $x^0 = 0$, and the first- and second-order derivatives, $f'(0) = 0$ and $f''(x) = 2 < 0$ for all x. As before, the value of e associated with x^0 is obtained from g^{-1}.

Now change the scale on which e is measured by replacing x in (2) with the transformation

$$t(x) = -\lambda^{-x}, \tag{3}$$

where $\lambda > 1$. Here t is defined on the set of all real numbers R but, in violation of one of the hypotheses of Proposition 1, it does not map onto R. Instead it maps R into the set of all negative real numbers. The transformation in (3) is allowable because e is only ordinally measured by x and all increasing transformations of scale are therefore admissible. And with $t'(x) = \lambda^{-x}\ln\lambda > 0$ everywhere, $t(x)$ is increasing. Of course, t would not be admissible were e cardinally or ratio measured. Upon substituting (3) into (2),

$$f \cdot t(x) = -\lambda^{-2x}. \tag{4}$$

From (4) it follows that $(f \cdot t)'(x) = 2\lambda^{-2}\ln\lambda$, which cannot be zero for any x. Thus, although a maximum exists at $x^0 = 0$ in (2), it disappears when the increasing transformation (3) is applied to x. Here, again, is a situation in which the question of which value of e maximizes output cannot be answered. That is because, depending on the scale employed, the maximizing value of x may not even exist. And there is no way to determine objectively whether the function f in (1) is capable of maximization.

Returning to the context of Proposition 1, in the case of differentiability, a stronger result is obtained.

Proposition 2. Under the hypotheses of Proposition 1, suppose that D is an open set and that both f and t are twice differentiable. Assume the derivative $t'(x) \neq 0$ on D. Then f has a maximum at x^0 with $f''(x^0) < 0$ in D if and only if the composite function $f \cdot t(x) = f(t(x))$ has a maximum at some $\bar{x}$ with $(f \cdot t)''(\bar{x}) < 0$ in D.

Proof: Suppose f has a maximum at x^0 in D and $f''(x^0) < 0$. Then according to Proposition 1, the composite function $f \cdot t(x) = f(t(x))$ has a maximum at, say, $\bar{x}$. It follows that $(f \cdot t)'(\bar{x}) = 0$ and $(f \cdot t)''(\bar{x}) \leq 0$. It remains to show that $(f \cdot t)''(\bar{x}) < 0$. Now for any x,

$$(f \cdot t)'(x) = f'(t(x))t'(x)$$

and

$$(f \cdot t)''(x) = f'\big(t(x)\big)t''(x) + f''\big(t(x)\big)[t'(x)]^2.$$

At the maximum $\bar{x}$,

$$(f \cdot t)''(\bar{x}) = f''(t(\bar{x}))[t'(\bar{x})]^2. \qquad (5)$$

Although the effect of applying the transformation t does not change the collection of function values of f, it can, as has been seen, change the maximizing value of x. It also can stretch or shrink parts of the domain of f and consequently alter the curvature of the function's graph. But since the function values have only been relocated over D and not altered, that stretching or shrinking cannot change the sign of the second-order derivative at the maximizing value as f is transformed into $f \cdot t$. Therefore $f''(t(\bar{x})) < 0$. It follows from (5) that $(f \cdot t)''(\bar{x}) < 0$.

QED

How the curvature of the graph of f changes when t operates on x depends on the nature of t. Consider (2) once again. Letting $t(x) = x + b$ for any b, then $f \cdot t(x) = -(x+b)^2$ and $(f \cdot t)''(x) = -2$, and although, as has been seen, the maximizing value changes to $\bar{x} = b$, the curvature of the graph remains unchanged. When $t(x) = ax$ for $a > 0$, then as before $f \cdot t(x) = -(ax)^2$ and for any x, the second order derivative $(f \cdot t)''(x) = -2a$. Thus, with no change in the zero-maximizing value, the curvature becomes greater everywhere if $a > 1$ and lesser everywhere if $a < 1$.

Note that although the conclusion of Proposition 1 applies to the transformation $t(x) = x^3$ (because t maps the real line onto itself), that of Proposition 2 does not since $t'(0) = 0$. In this case, the maximum of $f \cdot t(x)$ remains at the origin but $(f \cdot t)''(0) = 0$.

Another way of stating the conclusion of the second part of Proposition 2 is that, even if $t'(x) = 0$ for some x in D, the strict convexity around the maximizing value x^0 does not change around the new maximum at $\bar{x}$ upon application of the transformation of scale t. But it is possible that the curvature of the graph of f can be altered in a region that does not contain the critical point from strictly concave to strictly convex by applying an appropriate t. To illustrate, let

$$f(x) = -x^2 + x \qquad (6)$$

on the domain of all real numbers. Then $f'(x) = -2x + 1$ and f has a maximum at $x^0 = 1/2$. Moreover, $f(x) = -2$ for all x so that f is strictly concave throughout its domain. Now apply the transformation $t(x) = x^3$ by replacing x in (6) by x^3. Then

$f \cdot t(x) = -x^6 + x^3$

and $(f \cdot t)'(x) = -6x^5 + 3x^2$

for all x. The maximum of $f \cdot t$ occurs at $\bar{x} = (1/2)^{1/3}$ which is different from x^0. But for any x, the second-order derivative $(f \cdot t)''(x) = -30x^4 + 6x$ can be positive or negative. In particular, $(f \cdot t)''(x) > 0$ where $x < (1/5)^{1/3}$, and $(f \cdot t)''(x) < 0$ where $x > (1/5)^{1/3}$. Thus, upon application of t, the graph of f changes from strictly concave to strictly convex over the interval $(-\infty < x < (1/5)^{1/3})$. Note that $(1/5)^{1/3} < (1/2)^{1/3}$ so that the maximizing value of $f \cdot t$ lies, as it must, in the region where the graph of $f \cdot t$ is strictly concave.

There is also a special case in which the x^0 maximizing f does not change under dilation transformations of scale. The function f is homogeneous of degree k when, for all x in D,

$$f(\lambda x) = \lambda^k f(x), \qquad (7)$$

for any $\lambda > 0$.

Proposition 3. Under the hypotheses of Proposition 2, if f is homogeneous of any degree k and if $t(x) = ax$ for any constant $a > 0$, Then f has a maximum at x^0 with $f''(x^0) < 0$ in D if and only if the composite function $f \cdot t(x) = f(ax)$ has a maximum at x^0 with $(f \cdot t)''(x^0) < 0$ in D.

Proof: Suppose f has a maximum at x^0 in D and $f''(x^0) < 0$. Then $f'(x^0) = 0$. Since f is homogeneous of degree k, by Euler's Theorem f' is homogeneous of degree $k-1$ and f'' is homogeneous of degree $k-2$. Then, setting $\lambda = a$, in (7) gives $f'(ax) = a^{k-1}f'(x)$ and $f''(ax) = a^{k-2}f''(x)$ for any x. Therefore $(f \cdot t)'(x^0) = f'(t(x^0))t'(x^0) = f'(ax^0)a = a^k f'(x^0) = 0$,

and from (5),

$(f \cdot t)''(x^0) = f''(ax^0)a^2 = a^k f''(x^0) < 0$,

implying that $f \cdot t(x) = f(ax)$ has a unique maximum at x^0.

QED

Under the conditions of Proposition 3, since the maximizing value, x^0, does not change under positive dilations $t(x) = ax$ for $a > 0$, the value of x^0 and hence e^0 can be uniquely determined. That is, if e is ratio measured and f is homogeneous of any degree, then admissible changes in scale do not change the outcome of the maximization process. As previously shown, the maximization of (2) with respect to x is an illustration.

The following results may be drawn from this discussion:

Conclusion A. If e is cardinally or ratio measured, the application of any admissible (i.e., linear or dilation) transformation of scale to e cannot alter the value of e that maximizes f (assuming it exists) as long as f is, when necessary, appropriately adjusted to account for the scale change.

Conclusion B. When e is only ordinally measured and hence all increasing transformations of scale are admissible, the actual value of e that maximizes a generally accepted function f, even if that value exists uniquely across some measurement scales with the appropriate adjustments in f, still cannot be determined as the definitive maximizer of f. That is, because the application of some transformations of scale may result in an outcome for which a maximizing value of x does not exist, and because no scale transformation is privileged over any of the others, it is not possible to assert that a maximizing value of e exists.

III

It should be pointed out that there are ways to circumvent the ordinality problem addressed in Conclusion B. On the one hand, it can be just assumed away by supposing that an acceptable f has been sufficiently precisely specified and that the changes in scale that cause the difficulty have been ruled out. But this approach is not very satisfying because, given the ordinality of x, the choice of f and the limitations imposed on scale transformations are arbitrary. And that arbitrariness leads to questions of relevance and cogency of the analysis. On the other hand, the ordinal measurement of e could be discarded and the structural content of the problem contained in f could be revised to accommodate e as an unquantified variable. One possibility is as follows:

Let ρ be a reflexive, transitive, and total relation defined on E which is not necessarily capable of ordinal representation. Consider the family F of subsets S_e of E characterized by $S_e = \{e' : e' \rho e \text{ and } e' \text{ is in } E\}$, for all e in E. It is clear that, since ρ is total, F has the finite intersection property.[8] Suppose the S_e are closed according a topology that has been specified on E, and that E itself is compact. It follows that there exists an e^0 that is contained in every S_e (Kelley [8, p. 136] and hence that $e^0 \rho e$ for all e in E. In the latter sense, e^0 is "maximal" with respect to ρ. Thus a relation can be maximized with respect to an ordinal variable without

[8] F has the finite intersection property when the intersection of every finite subfamily of F is nonempty.

relying on its ordinality in any way.

In the next section an example is presented from the Economics literature in which a function is maximized with respect to a variable which might be rejected as cardinally or ratio measured. Given that rejection, the most that can be hoped for is ordinal measurement.

IV

The main purpose of economic analysis is taken here to be the explanation of observable economic phenomena. Explanatory models of observable economic behavior typically set up a framework within which decisions are taken that propel economic behavior. Assumptions are then imposed that drive the action of making the decision, thereby explaining the behavior. An example will be given here in which optimization is the decision-making force that generates behavior. In this case, the cogency and viability of the explanation depends on whether certain variables in the models can be measured at least on a cardinal scale.

Consider the attempt by Becker and Lewis [1] to explain an observed interaction between the number of children a family has and the quality of those children. Becker and Lewis begin with a family's ordinal utility function

$$\mu = u(n, q, z),$$

defined on $\{(n, q, z): n \geq 0, q \geq 0, z \geq 0\}$, where n is the number of children, q is their measured quality (assumed to be identical across all children), and z is the number of units of consumption commodities (appropriately condensed into one) consumed. This function represents the family's ordered preferences among vectors (n, q, z). Since children of any quality and consumption are costly, the family is constrained by its income I:

$$I = p_c nq + p_z z, \qquad (8)$$

where $p_c > 0$ and $p_z > 0$ are parameters representing, respectively, the price per unit of nq (which may be thought of as "total" child quality) and the price per unit of z. The family choice of (n, q, z) is thought to be determined by maximizing u subject to the constraint (8). Assuming that unique global maxima exist for all $p_c > 0$ and $p_z > 0$ and that u has sufficient differentiability properties, the method of Lagrange multipliers is used to locate each maximum. The Lagrangean expression is

$$L(n, q, z) = u(n, q, z) - \lambda(p_c nq + p_z z - I),$$

where λ is the appropriate multiplier. Differentiating L with respect to the three variables, setting those derivatives equal to zero, and denoting the partial derivatives of u by subscripts,

$$u_n(n,q,z) = \lambda p_c q,$$
$$u_q(n,q,z) = \lambda p_c n,$$
and
$$u_z(n,q,z) = \lambda p_z.$$

These three equations together with (8) determine values for n, q, and z given p_c and p_z.

Becker and Lewis employ shadow prices of n, q and z in much of their discussion about the interaction between the determined values of child quantity n and measured child quality q. But it is not necessary to examine their argument further. The point of interest here is the calibration of child quality. How is that to be measured — as a child's intelligence indicated by a performance on an IQ test, or as money spent on his/her education, or something else? According to Gould [3, Ch. 5], IQ tests do not actually measure intelligence.[9] And the spending of money on education does not necessarily result in an educated person. More generally, these latter measures, although ratio in nature, do not seem to include enough characteristics of what might be referred to as child quality. Many elements such as health, energy, emotional stability, ability to interact with others, motivation to succeed, etc. are left out. However, it may still be possible to describe in words various manifestations of child quality in terms of these characteristics, and order those manifestations according to perceived child quality. In this way, an ordinal measure of child quality might be obtained. But, given the descriptions of cardinal and ration measures detailed at the outset, it is hard to see how either could be procured as a measure of child quality. And if cardinal and ratio scales are ruled out and ordinality is all that is attainable, then, as described above, the maximization of u subject to constraint is unable to determine a unique child quality, and the Becker-Lewis argument breaks down.

Conclusion.

This paper deals with the permissible transformations of the scale of measurement applied to the variables with respect to which that function is optimized. The collection of permissible transformations depends on whether the variable is measured on a ratio, cardinal (interval), or ordinal scale.

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9 Apparently, the inventor of the IQ test, Alfred Binet, developed the test for different purposes and did not believe it measured intelligence, (Gould [3, p. 151]). "If Binet's principles had been followed, and his tests consistently used as he intended, we would have been spared a major misuse of science in our century." Gould [3, p. 155].

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